Total Search Problems in Game Theory and Economics

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Alexandros Hollender, University of Oxford, EPFL

Tutorial SAGT 2022

Focus on PPAD

Some PPAD-complete Problems

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- NASH
  - $k \geq 2$ players [Daskalakis-Goldberg-Papadimitriou 09, Chen-Deng-Teng 09]

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Focus on PPAD

I. How do we prove PPAD-membership?

II. How do we prove PPAD-hardness?

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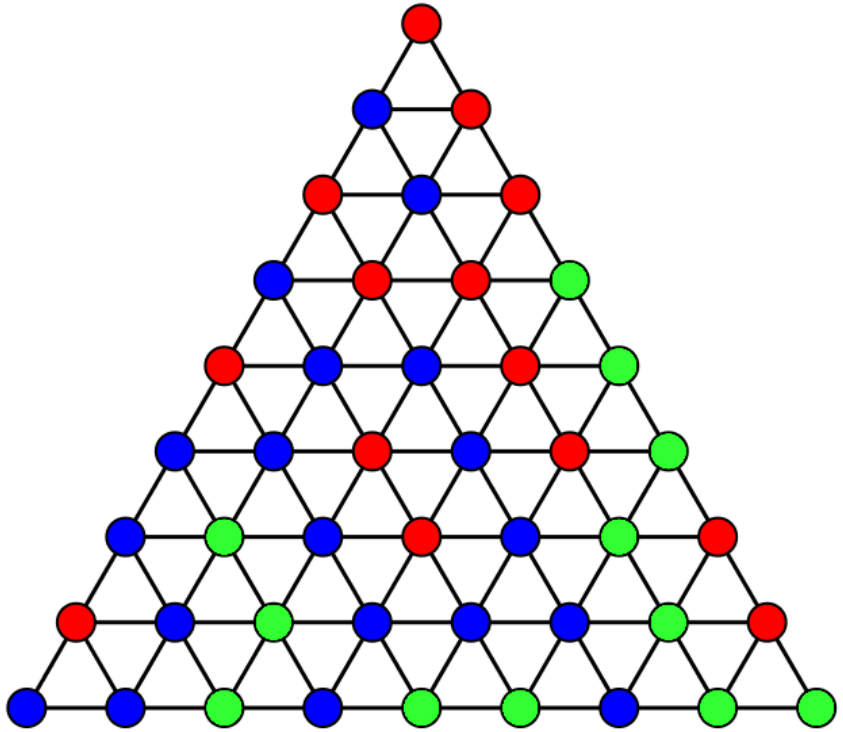
I. How do we prove PPAD-membership?
   1. **Sperner's Lemma**

II. How do we prove PPAD-hardness?

Sperner's Lemma

Sperner's Lemma (Sperner 1928):

Triangulation of a big triangle.



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Sperner coloring (3 colors):

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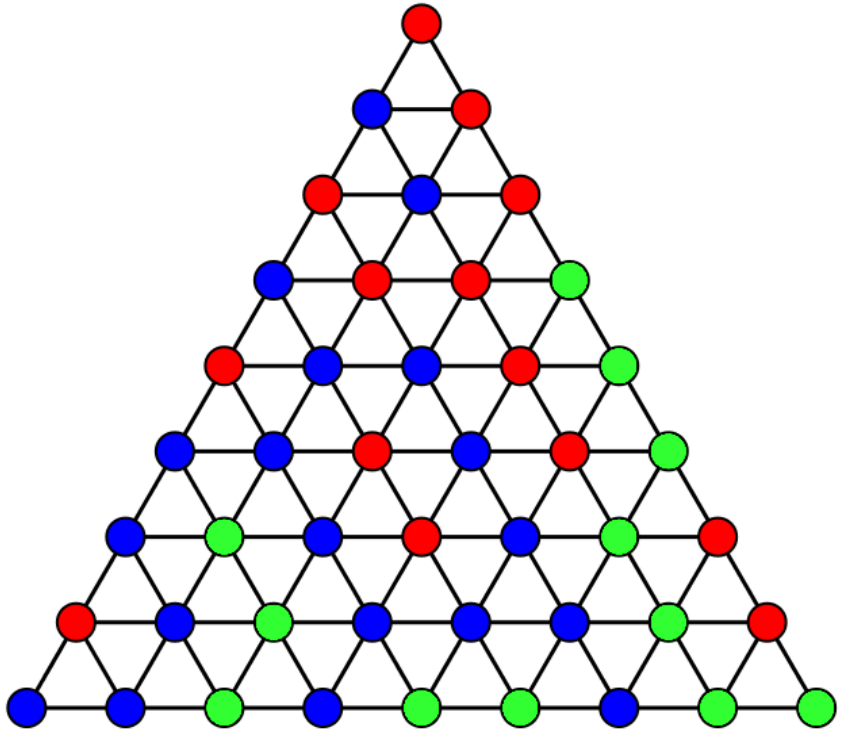

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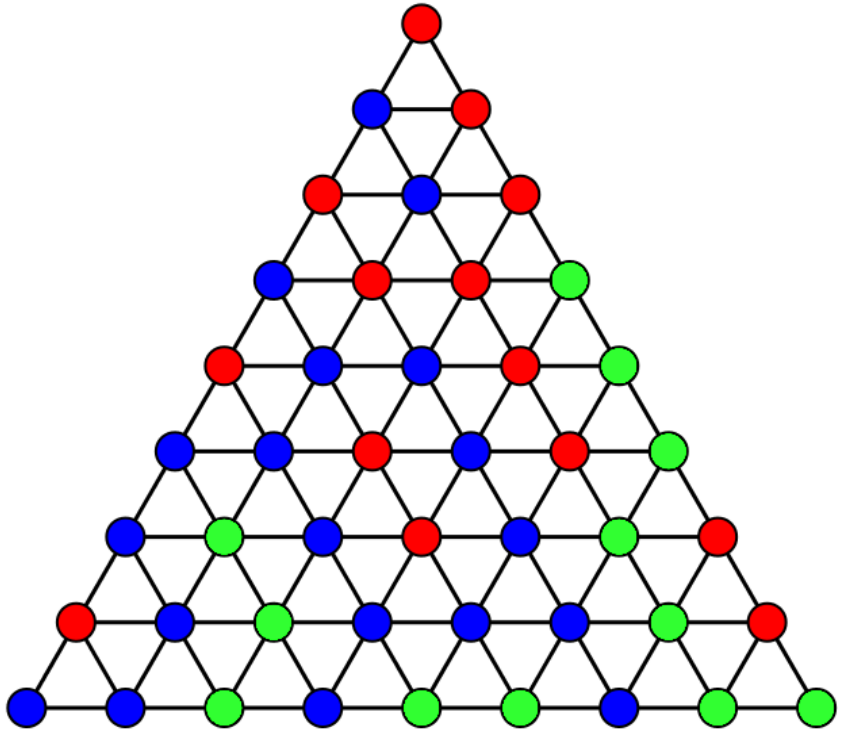

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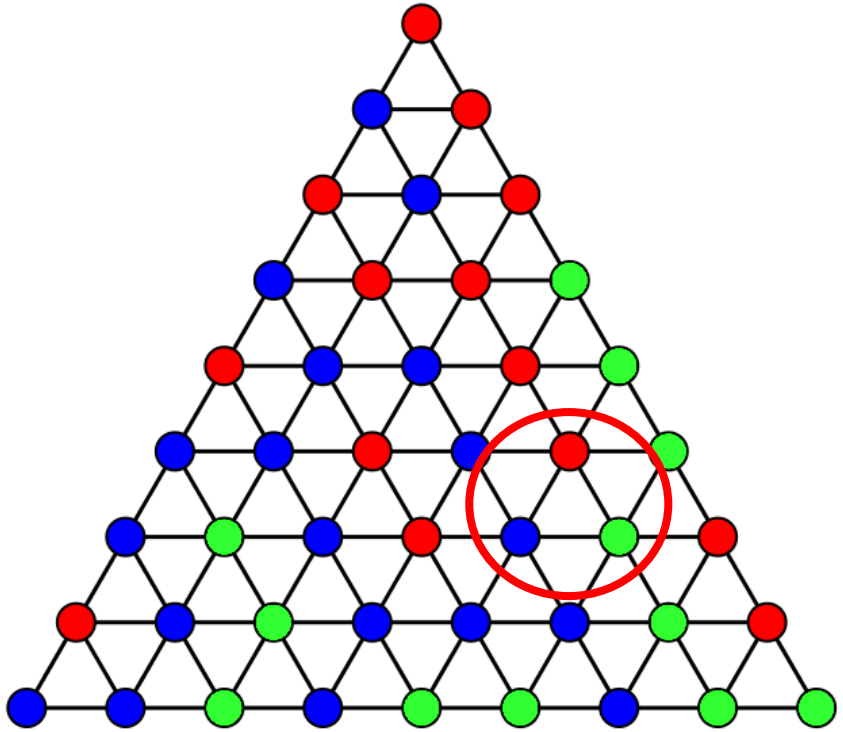

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Triangulation of a d-dimensional simplex.

Sperner coloring ($d + 1$ colors):
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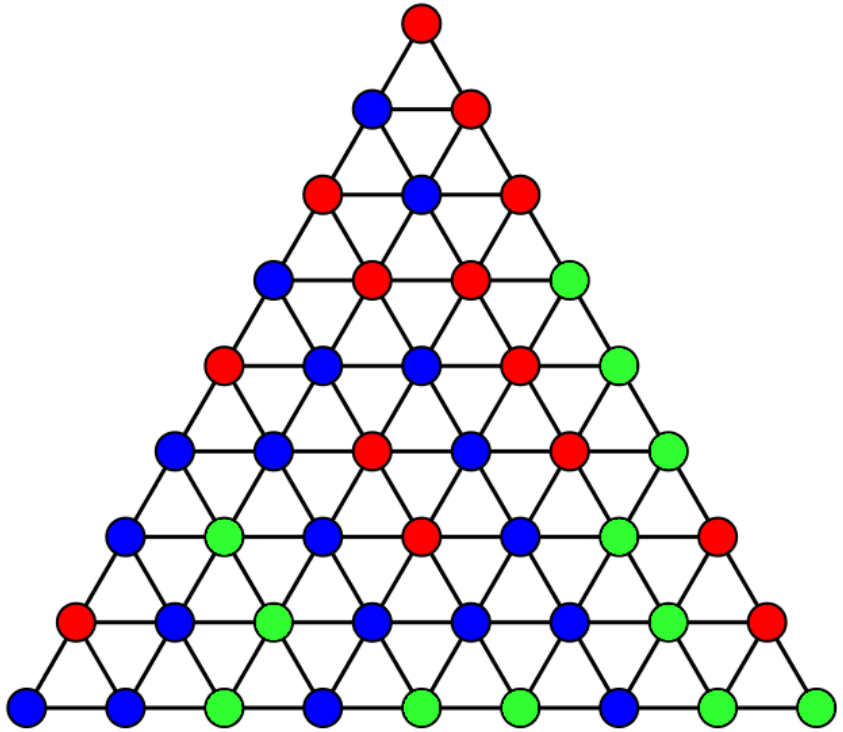
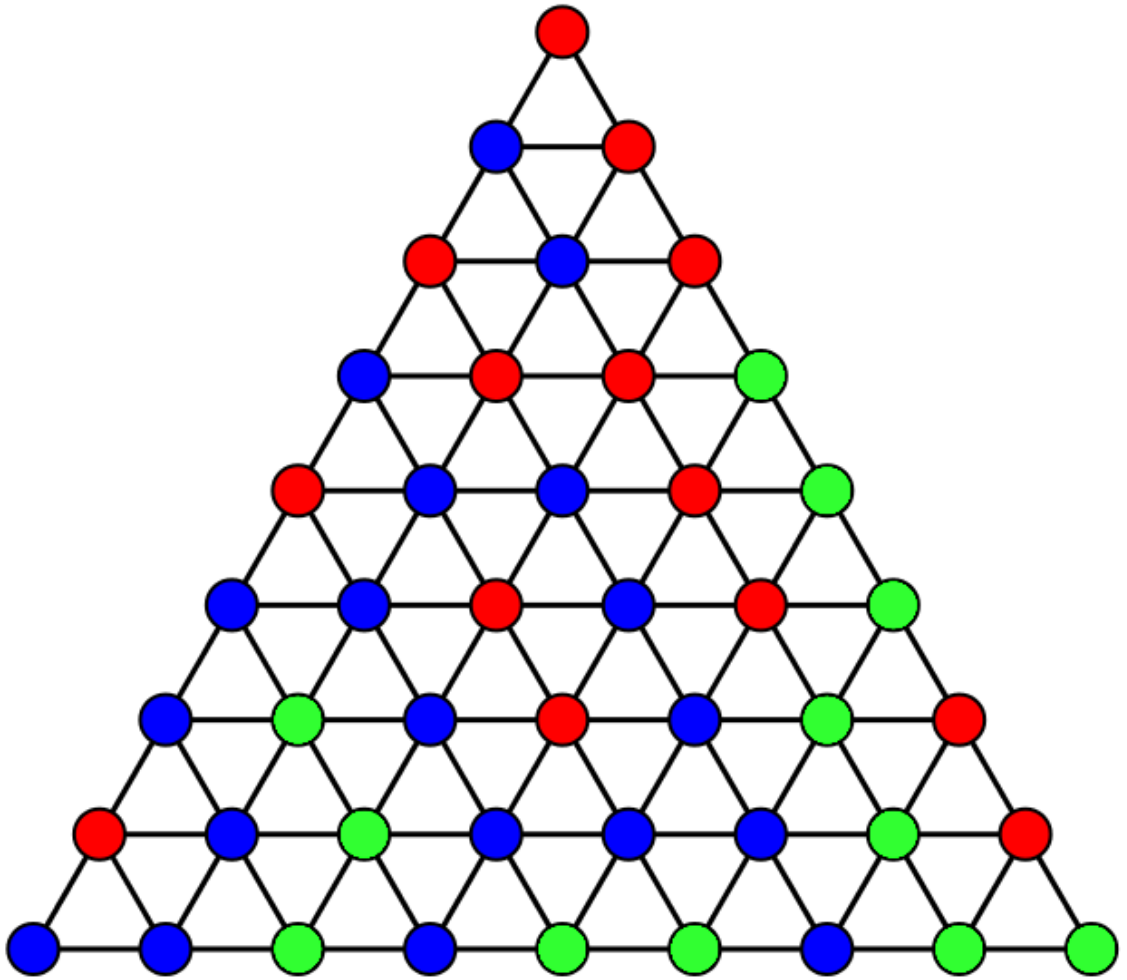


→ There always exists a panchromatic small simplex.

A Proof of Sperner's Lemma



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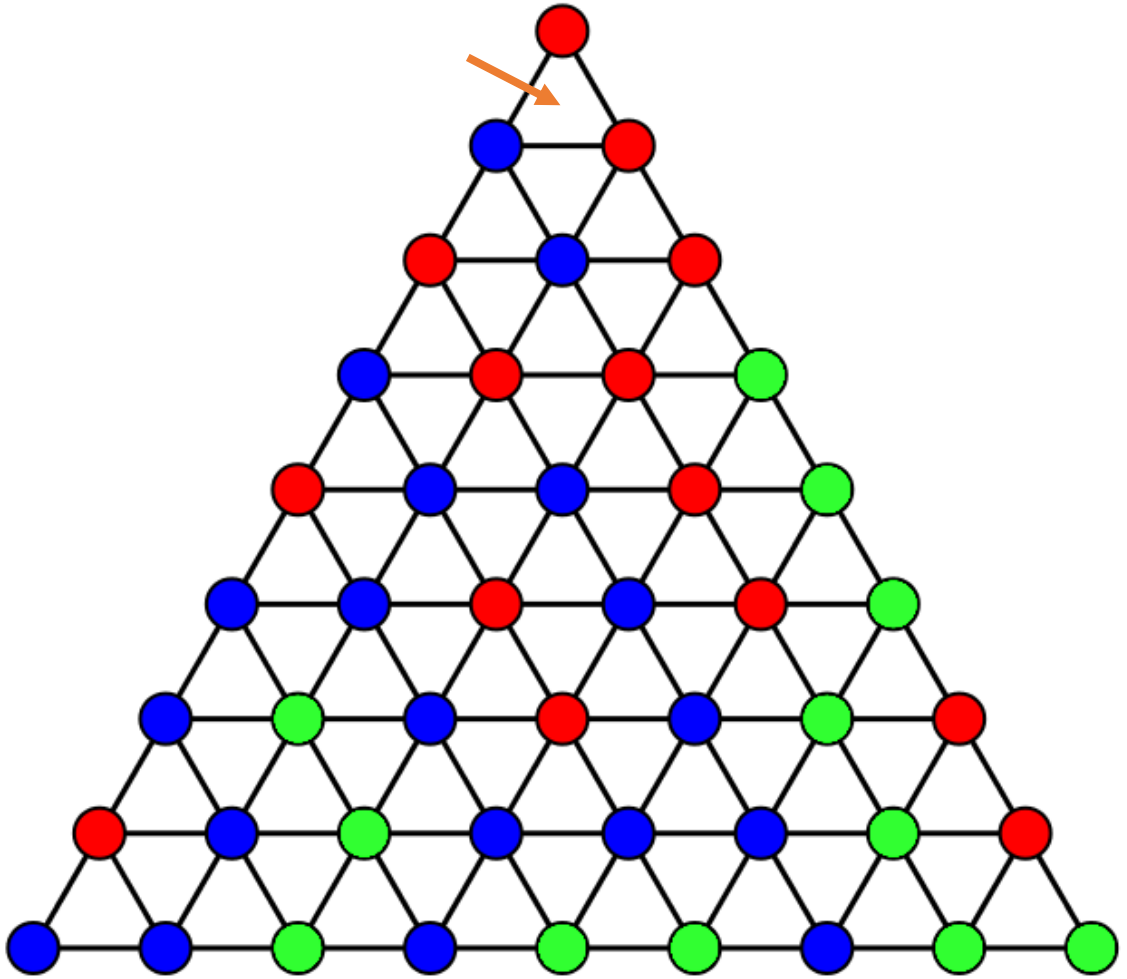

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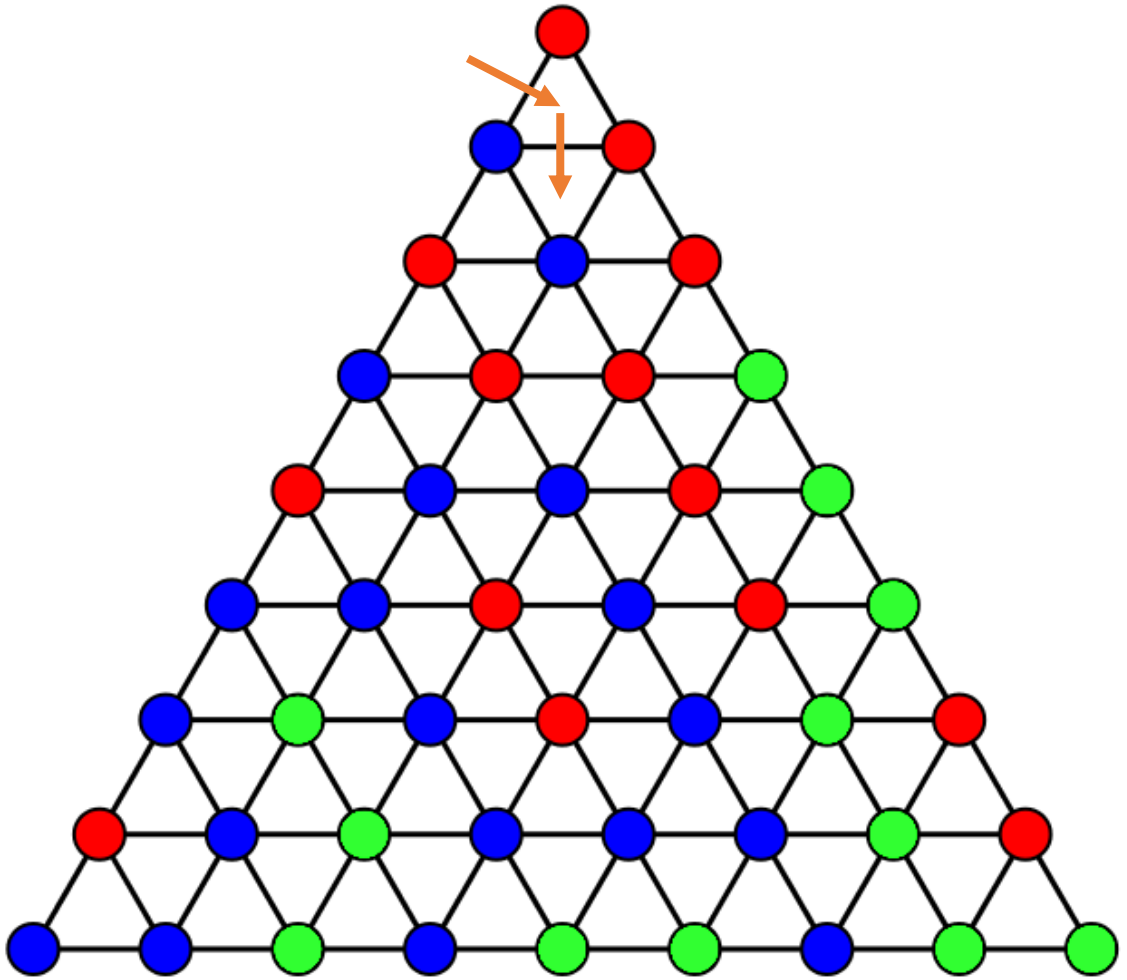

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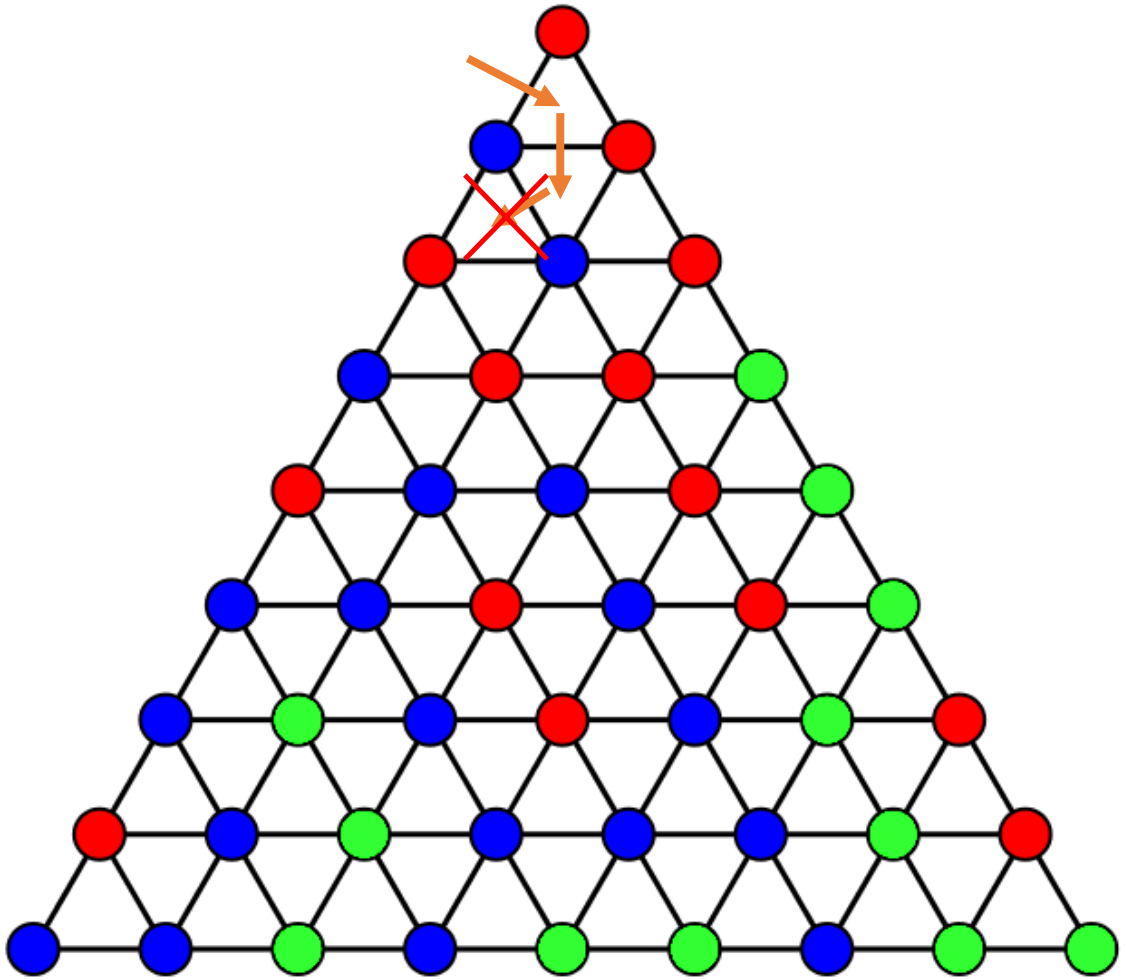

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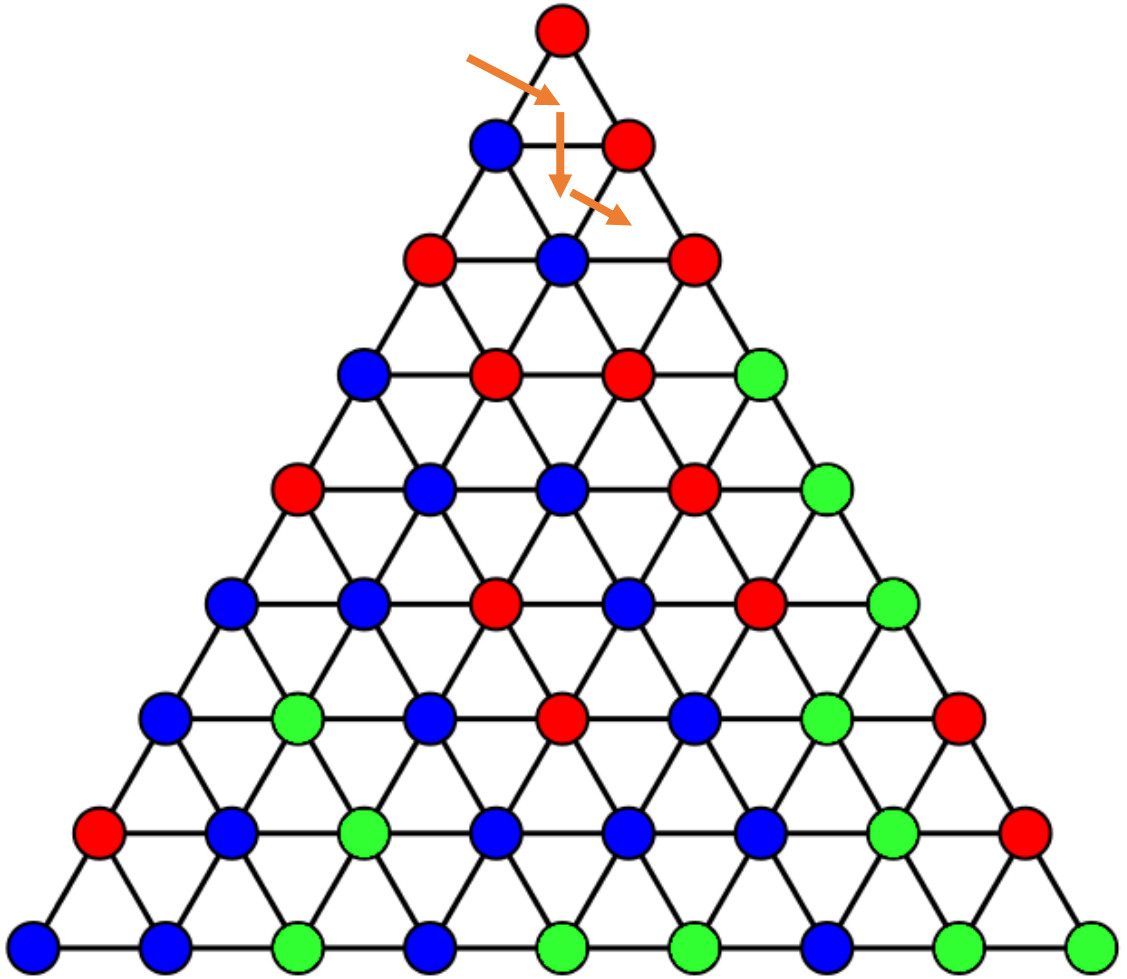

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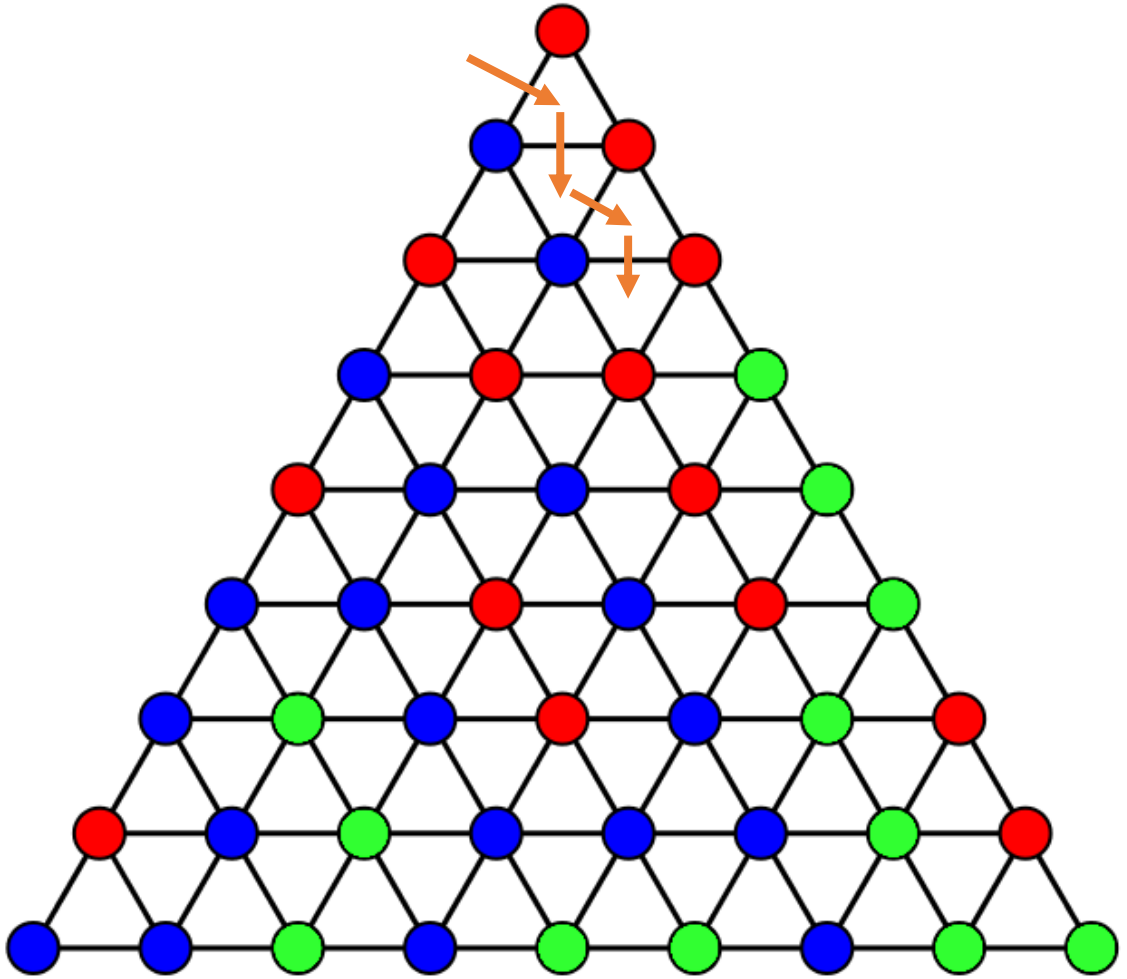

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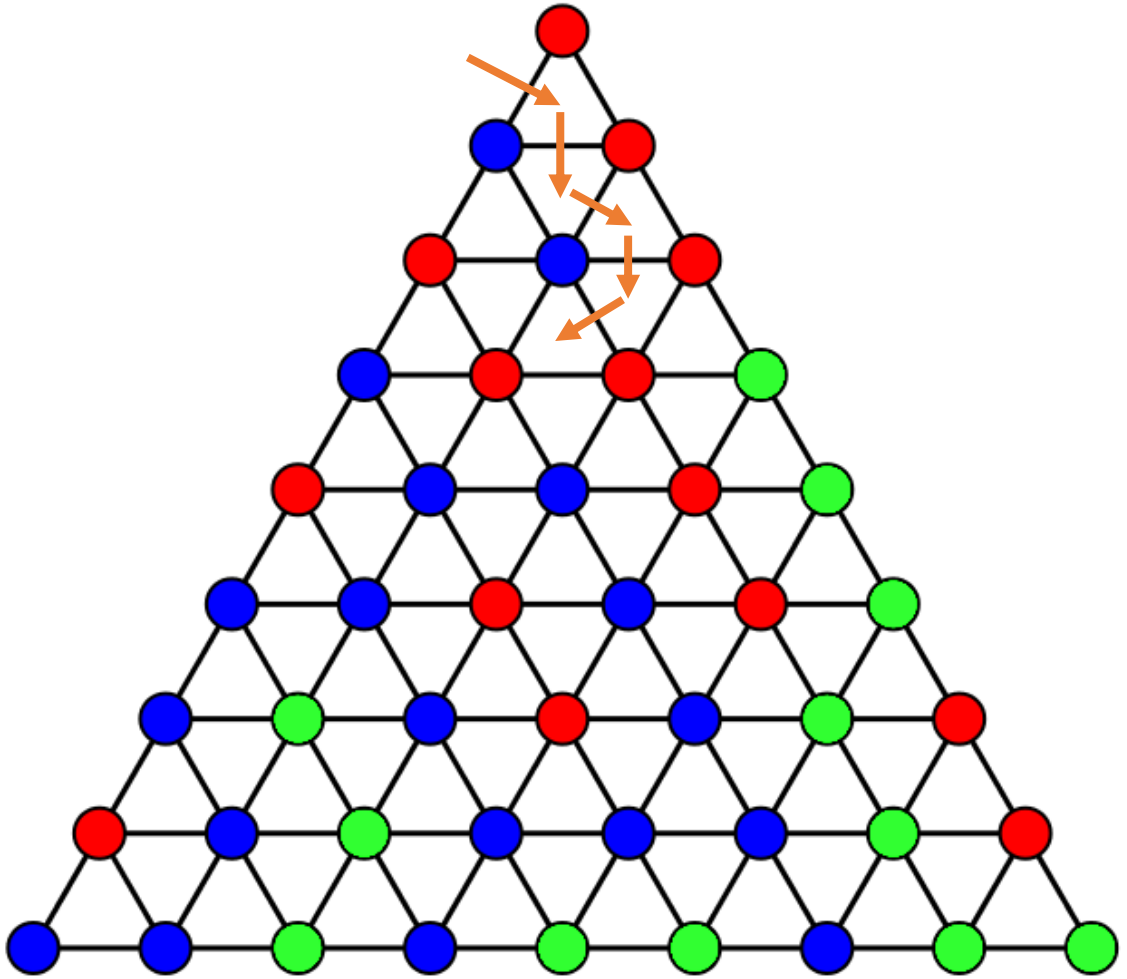

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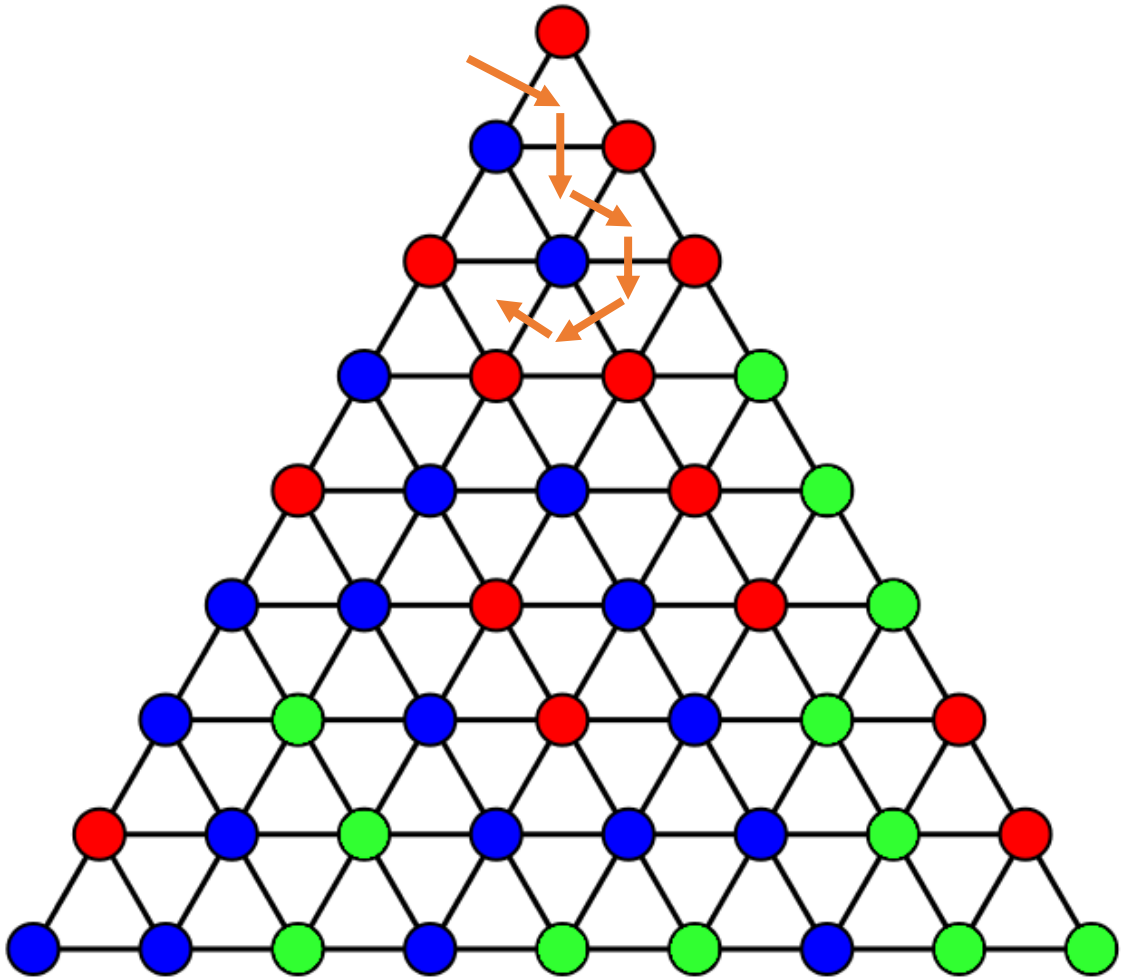

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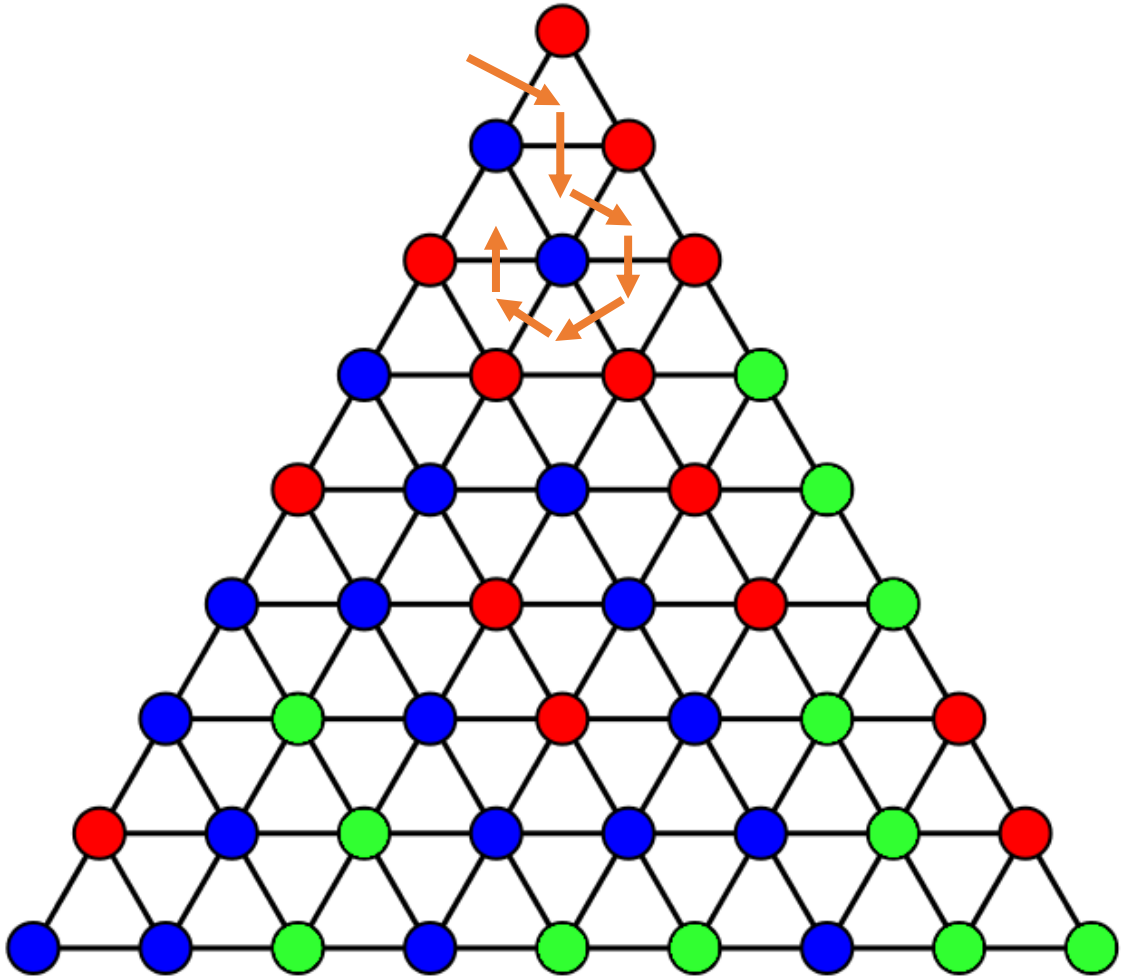

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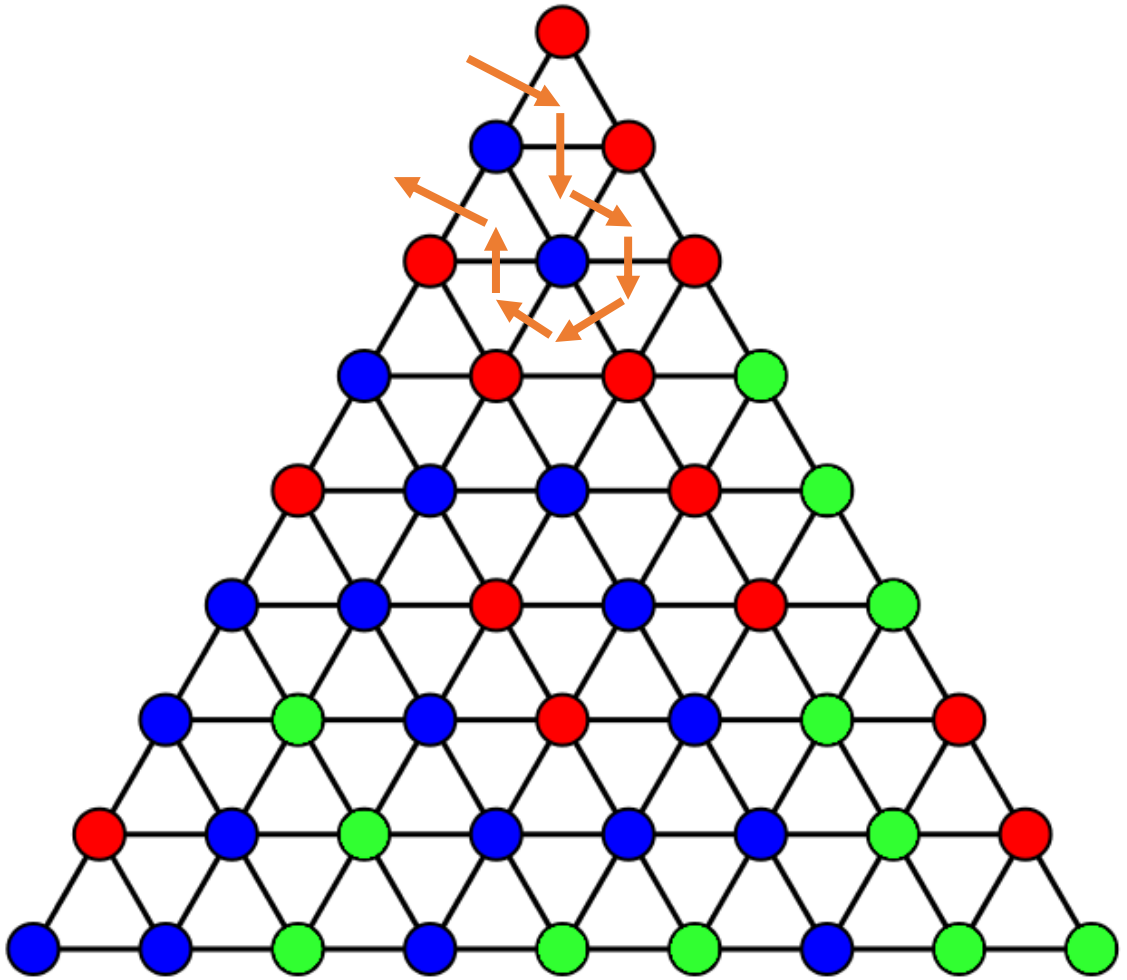

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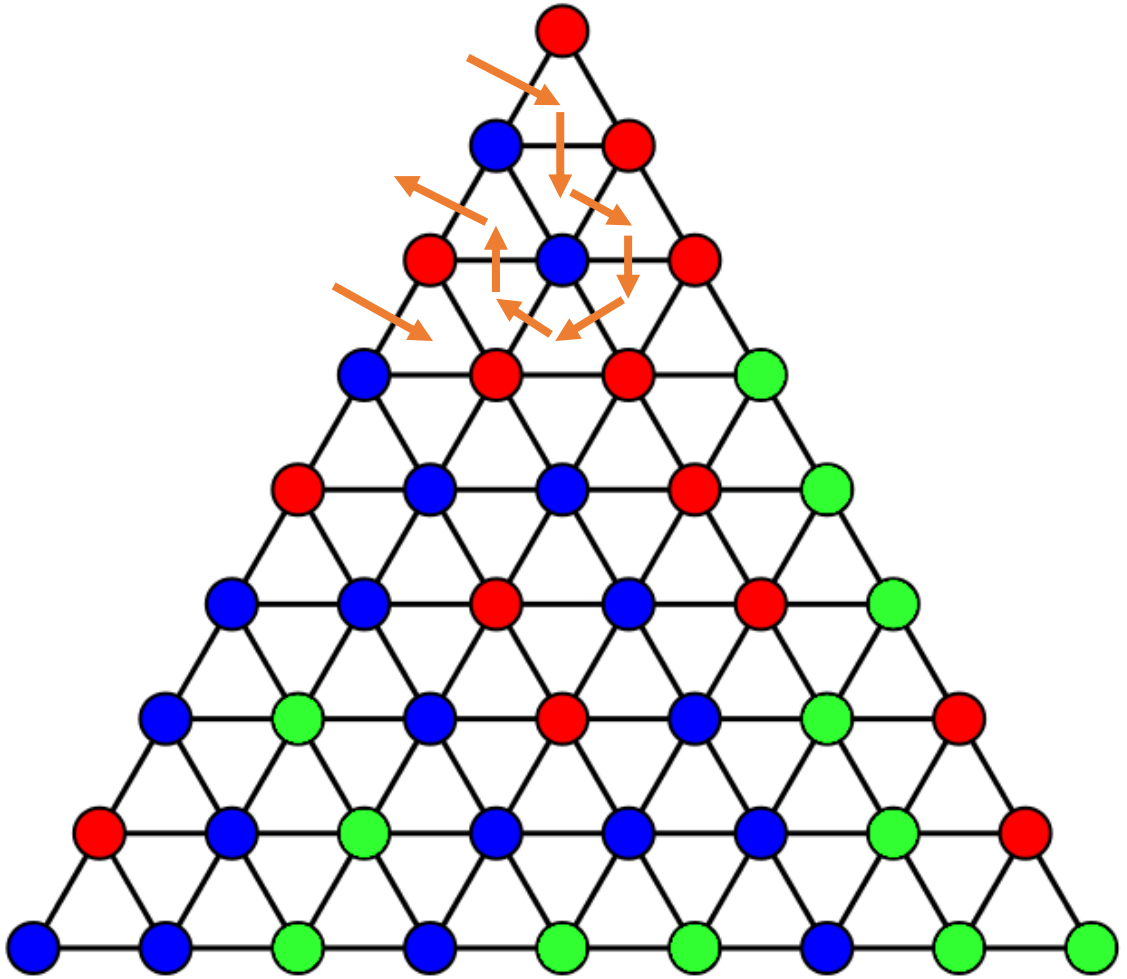

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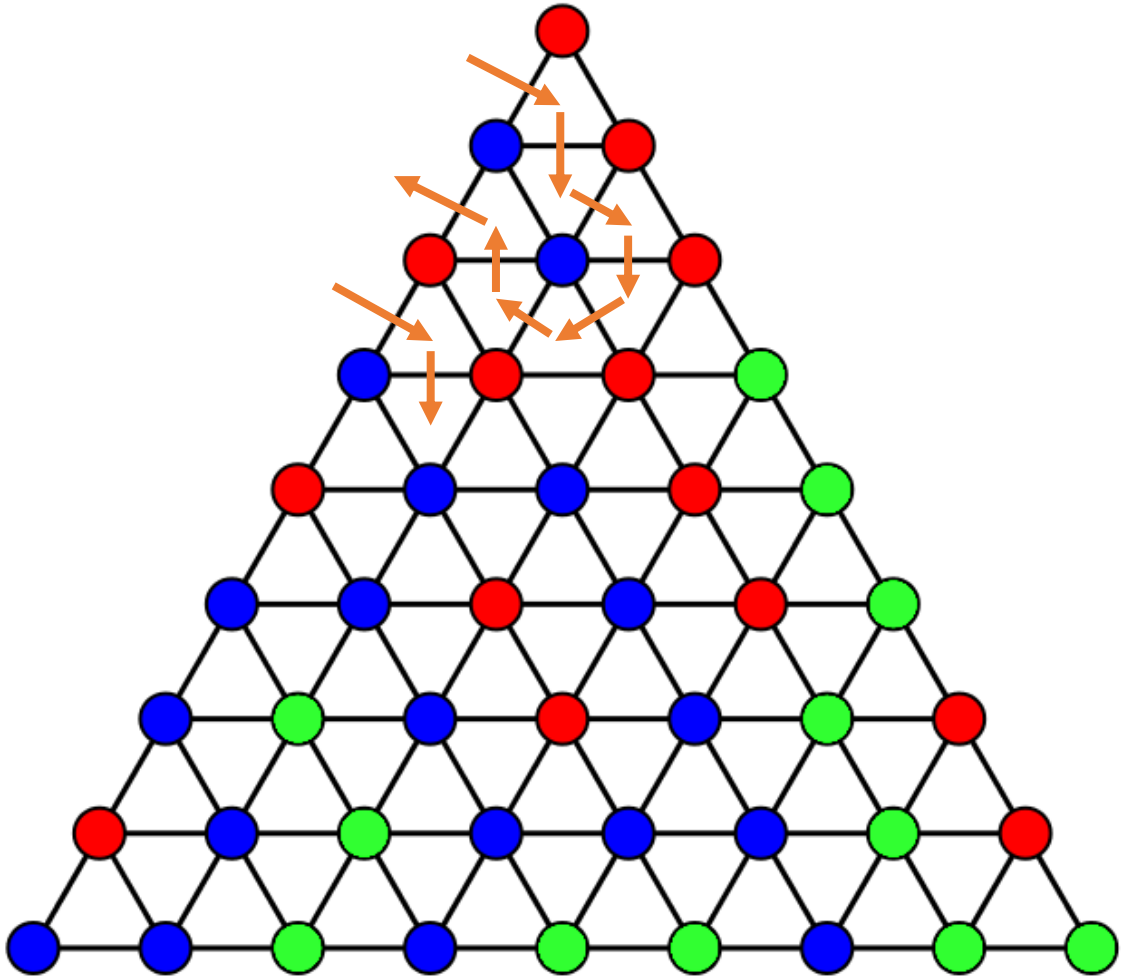

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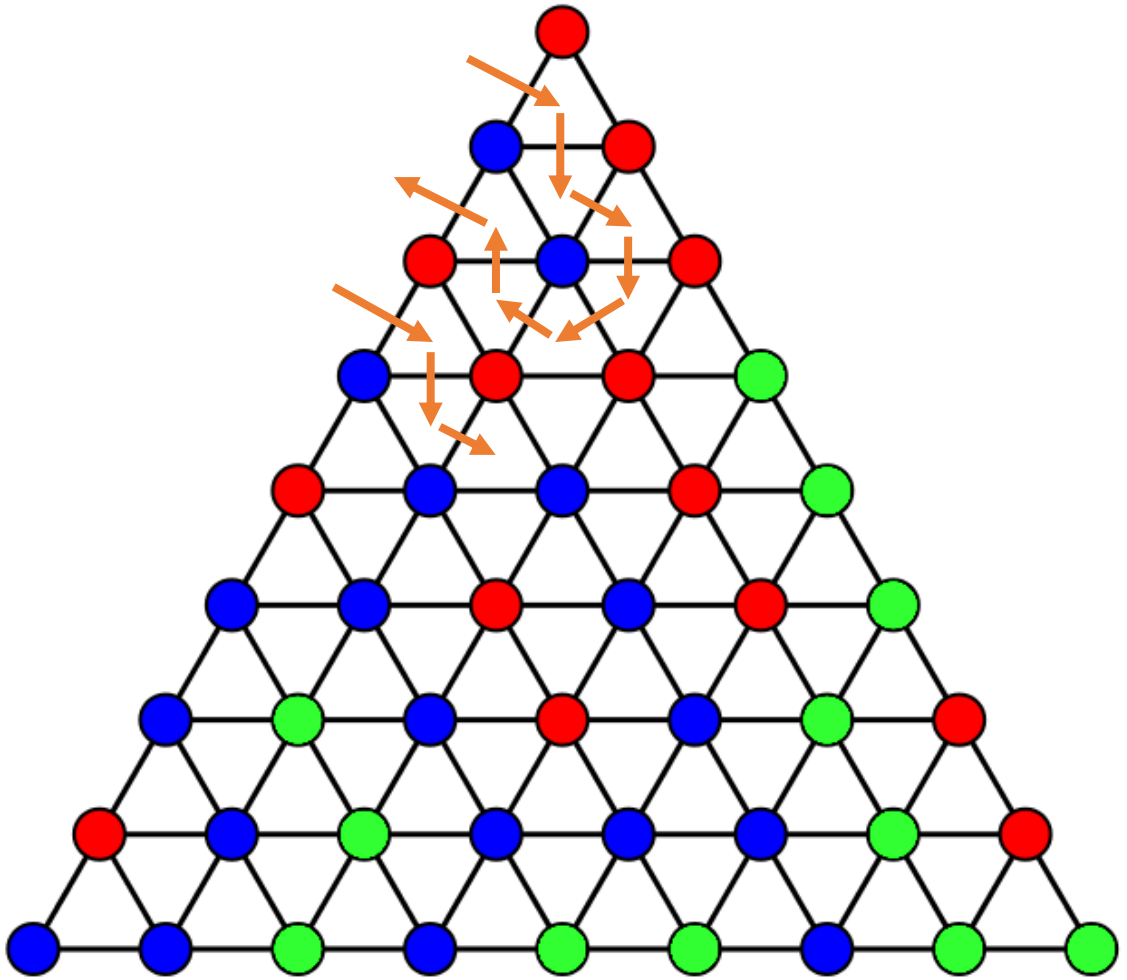

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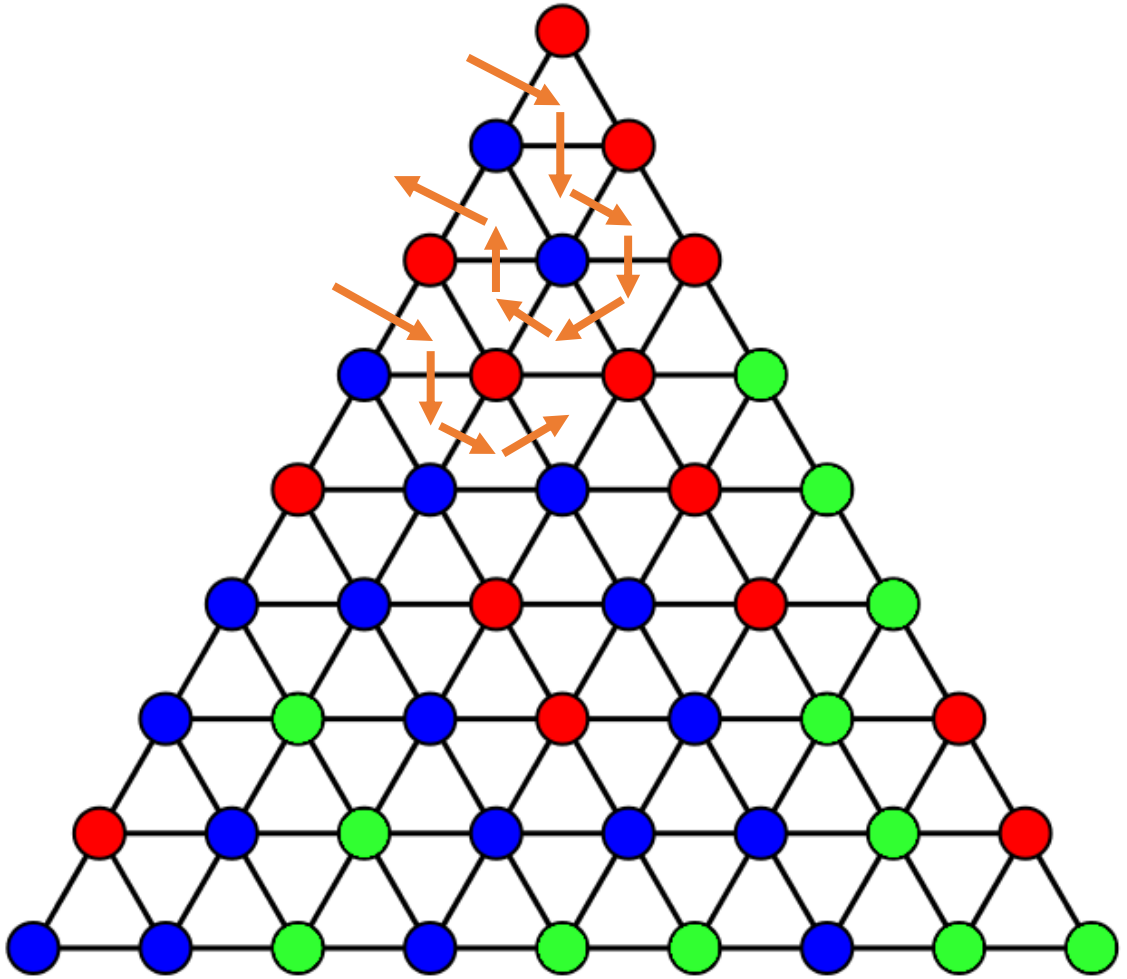

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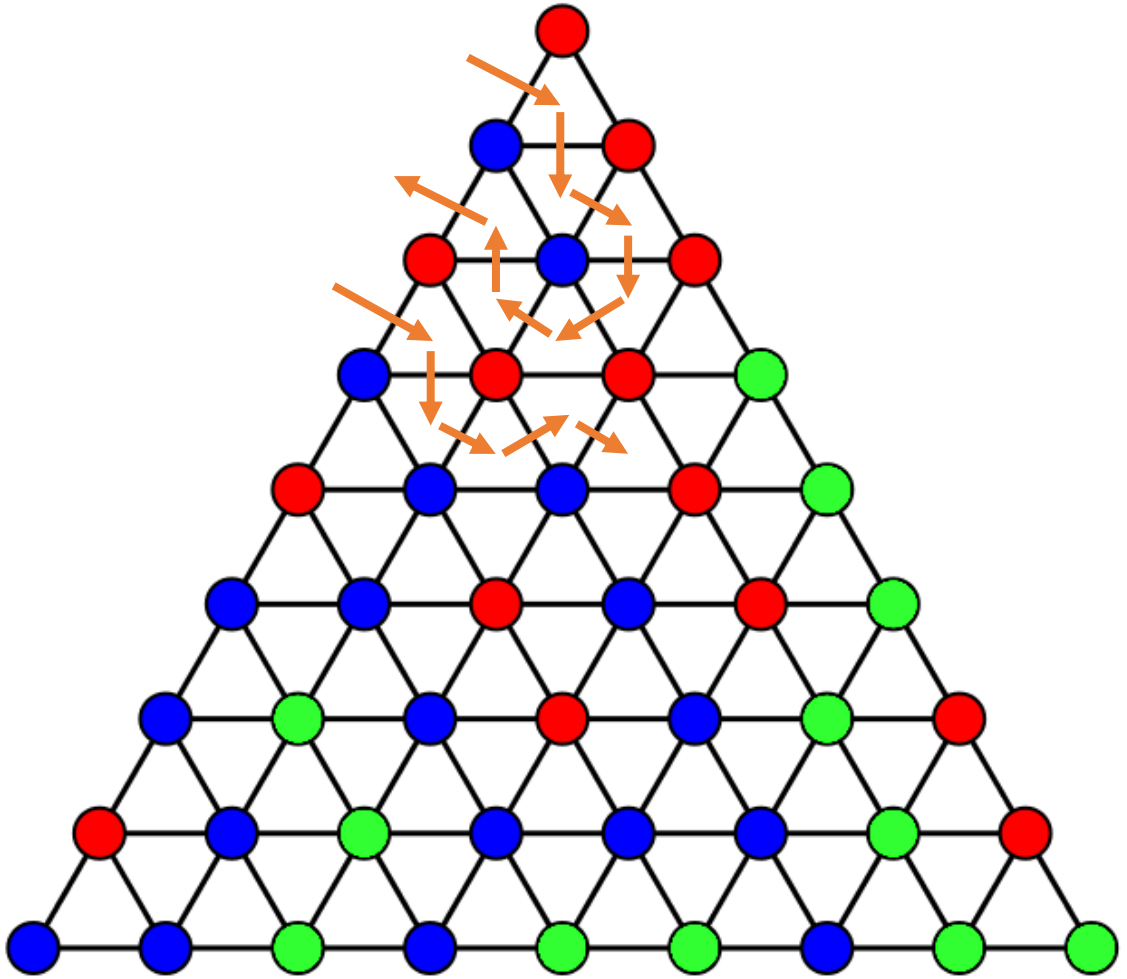

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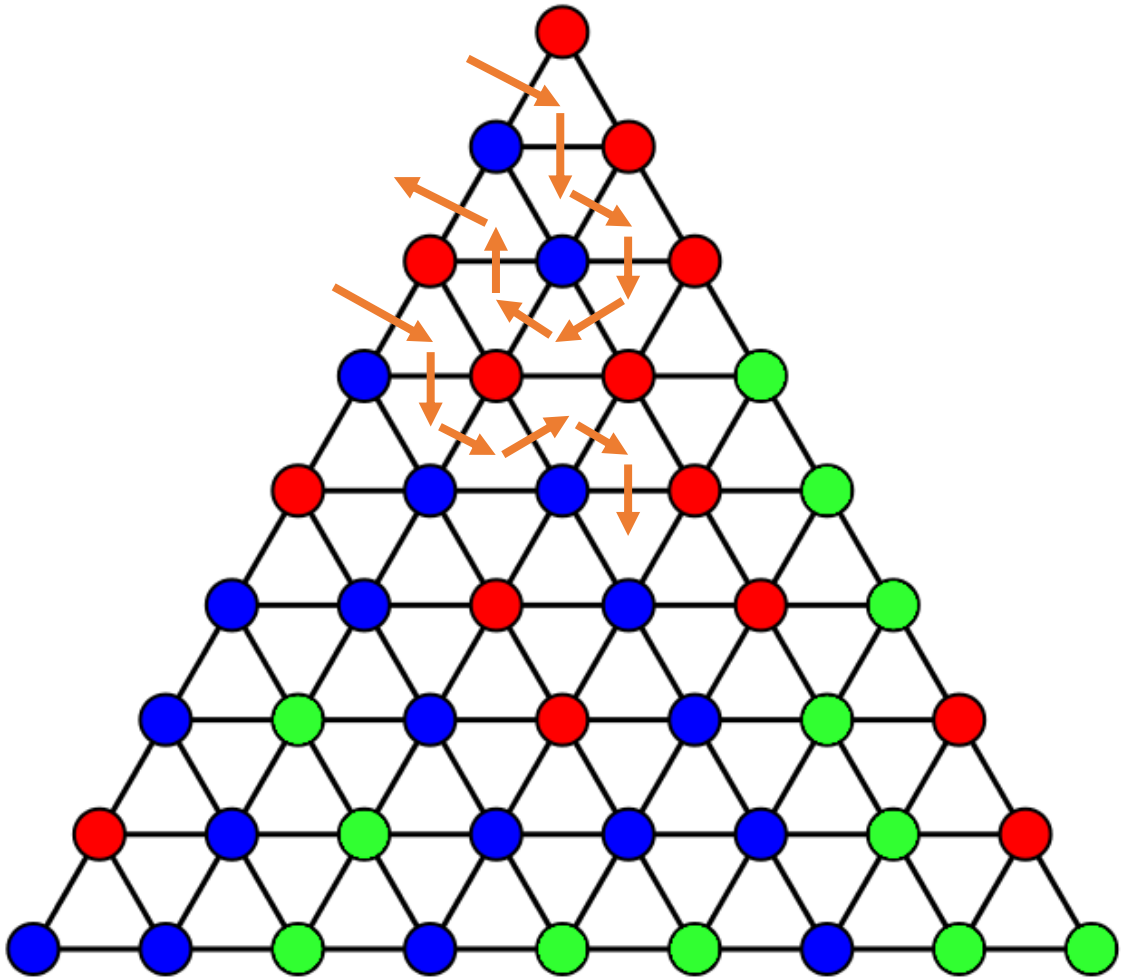

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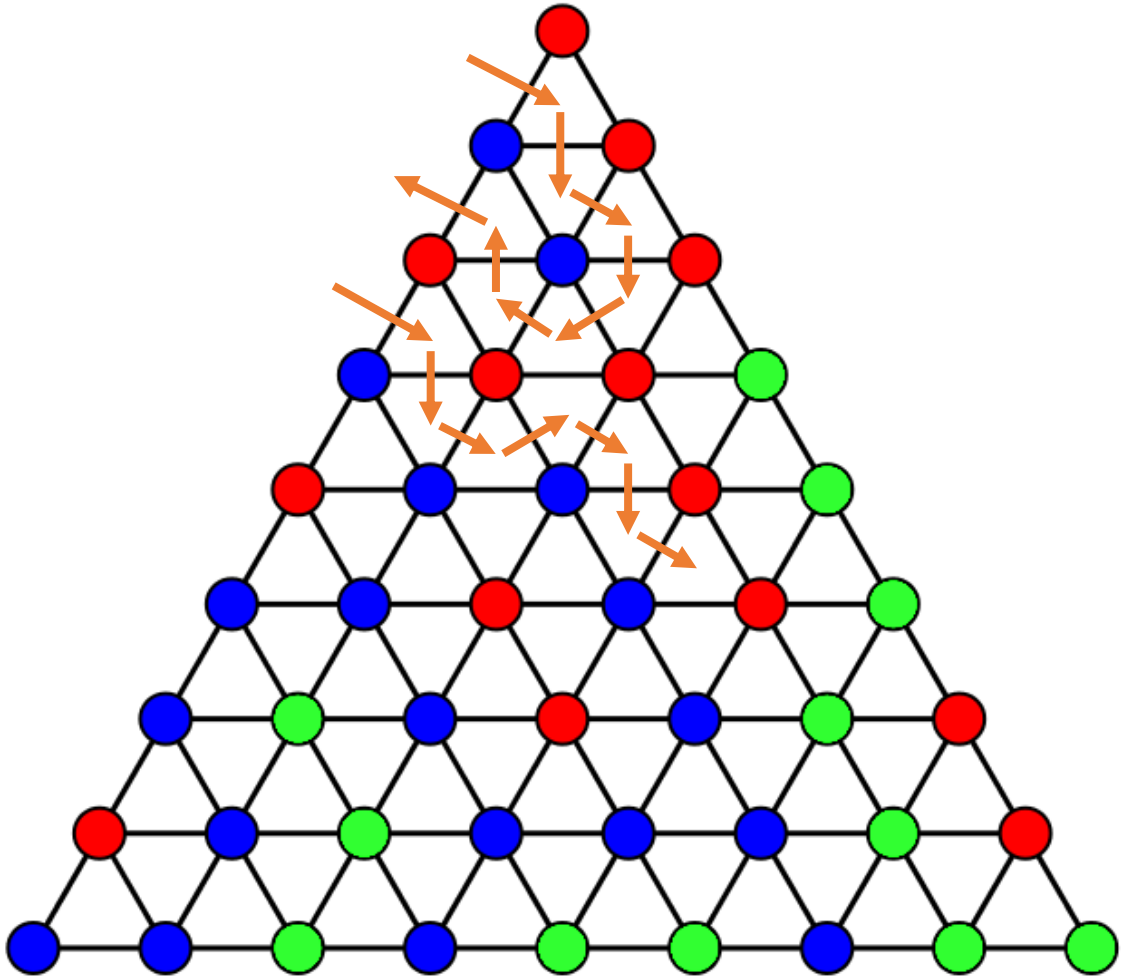

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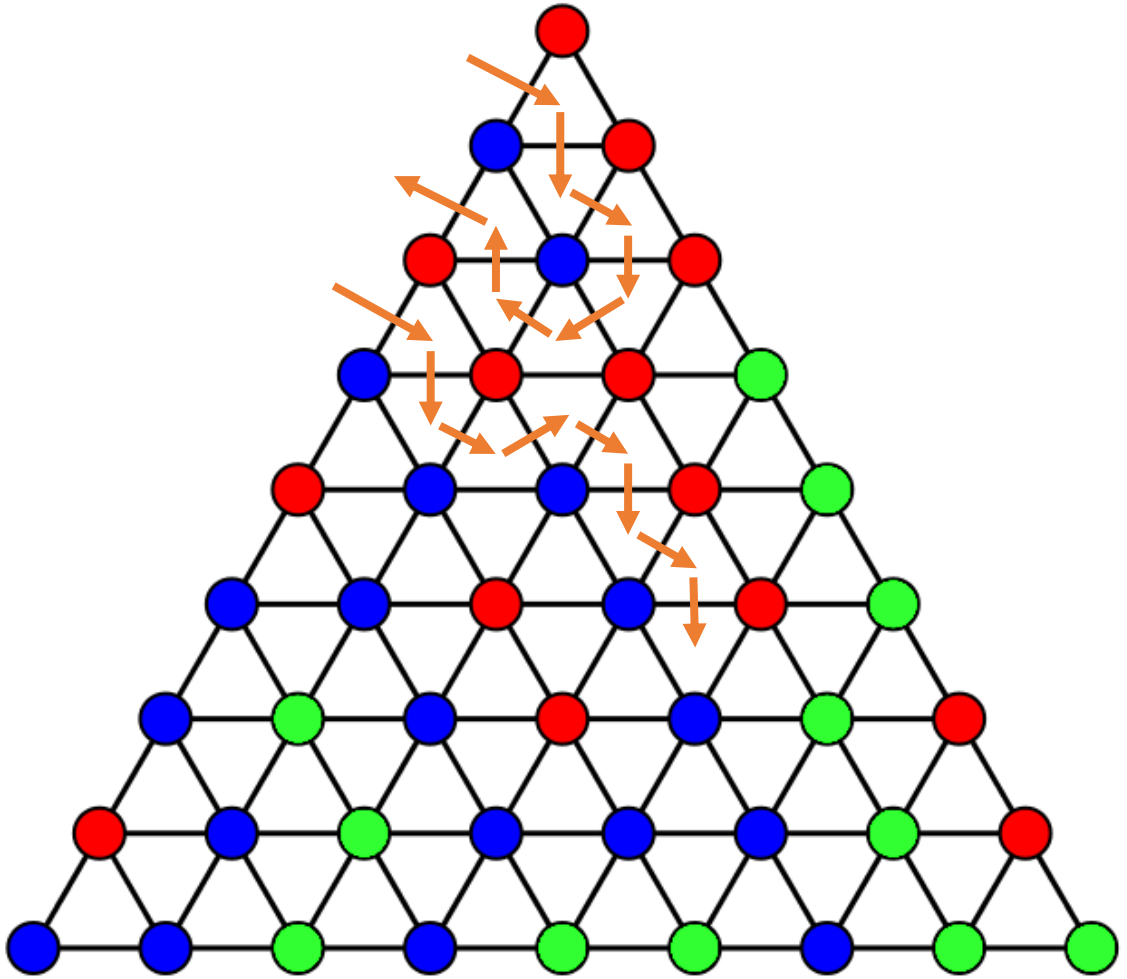

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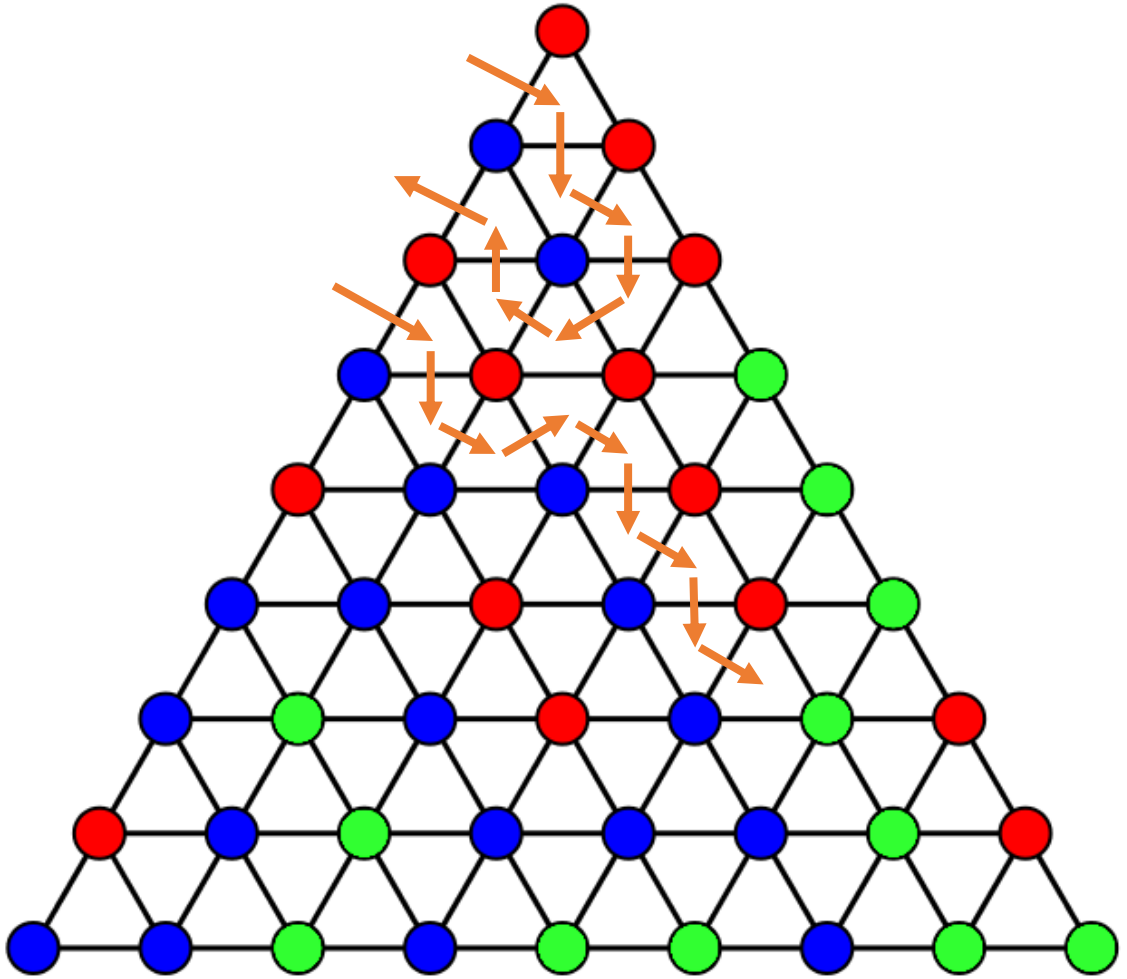

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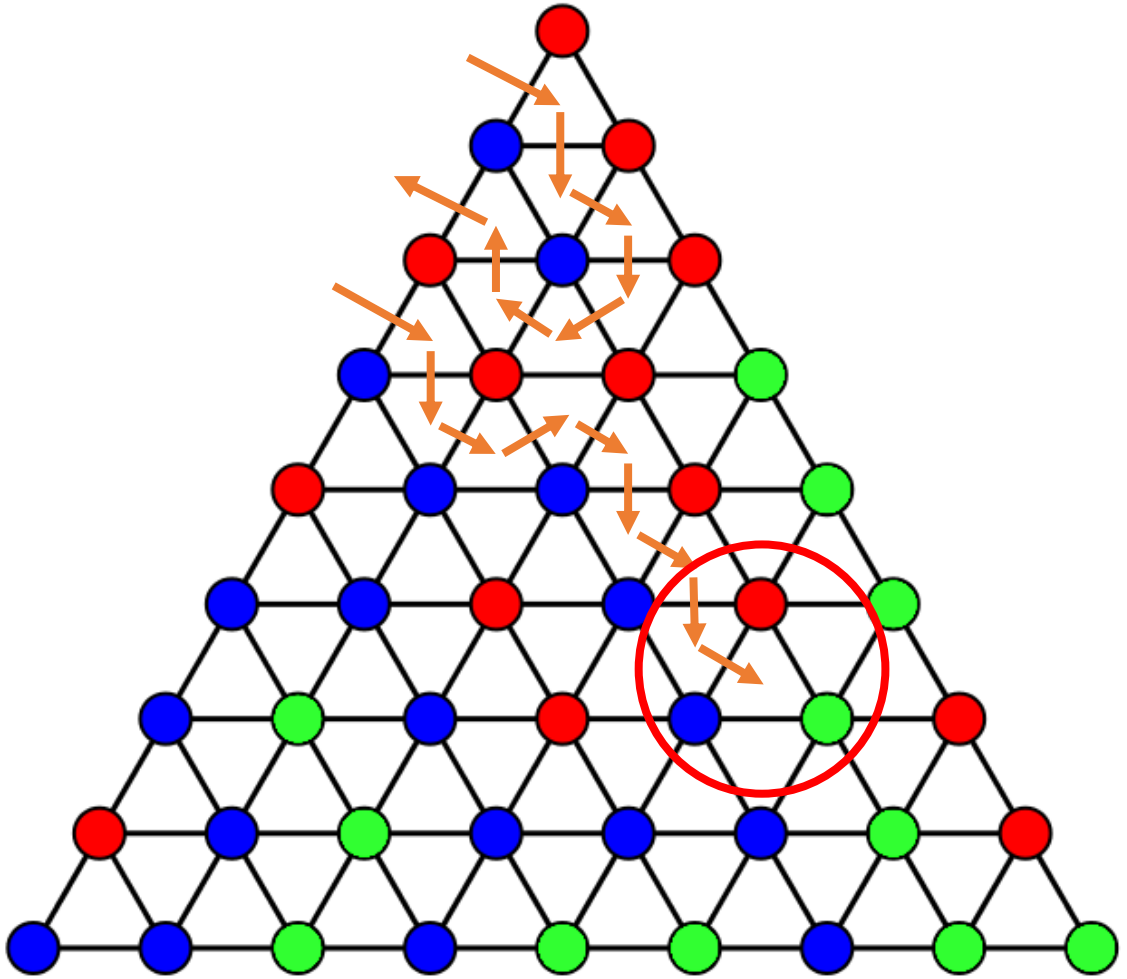

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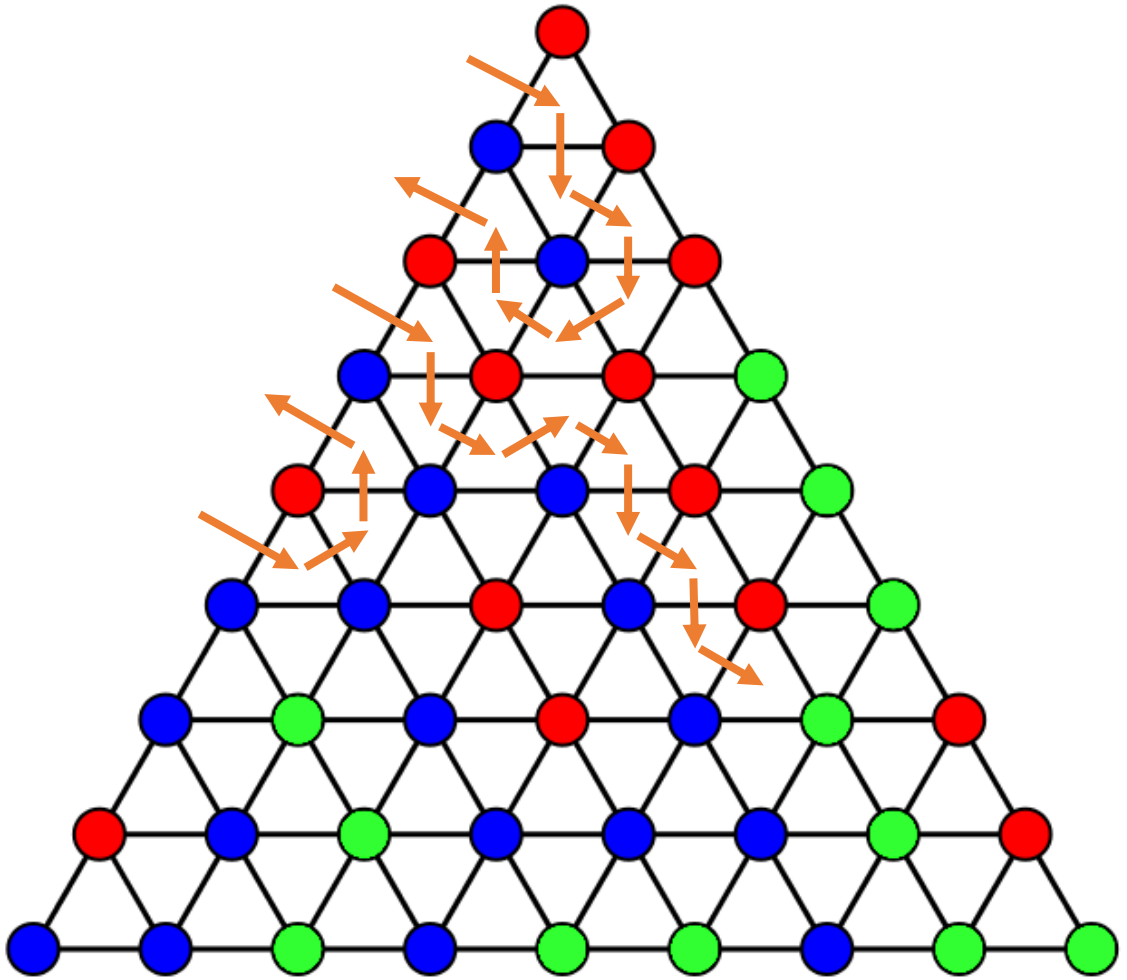

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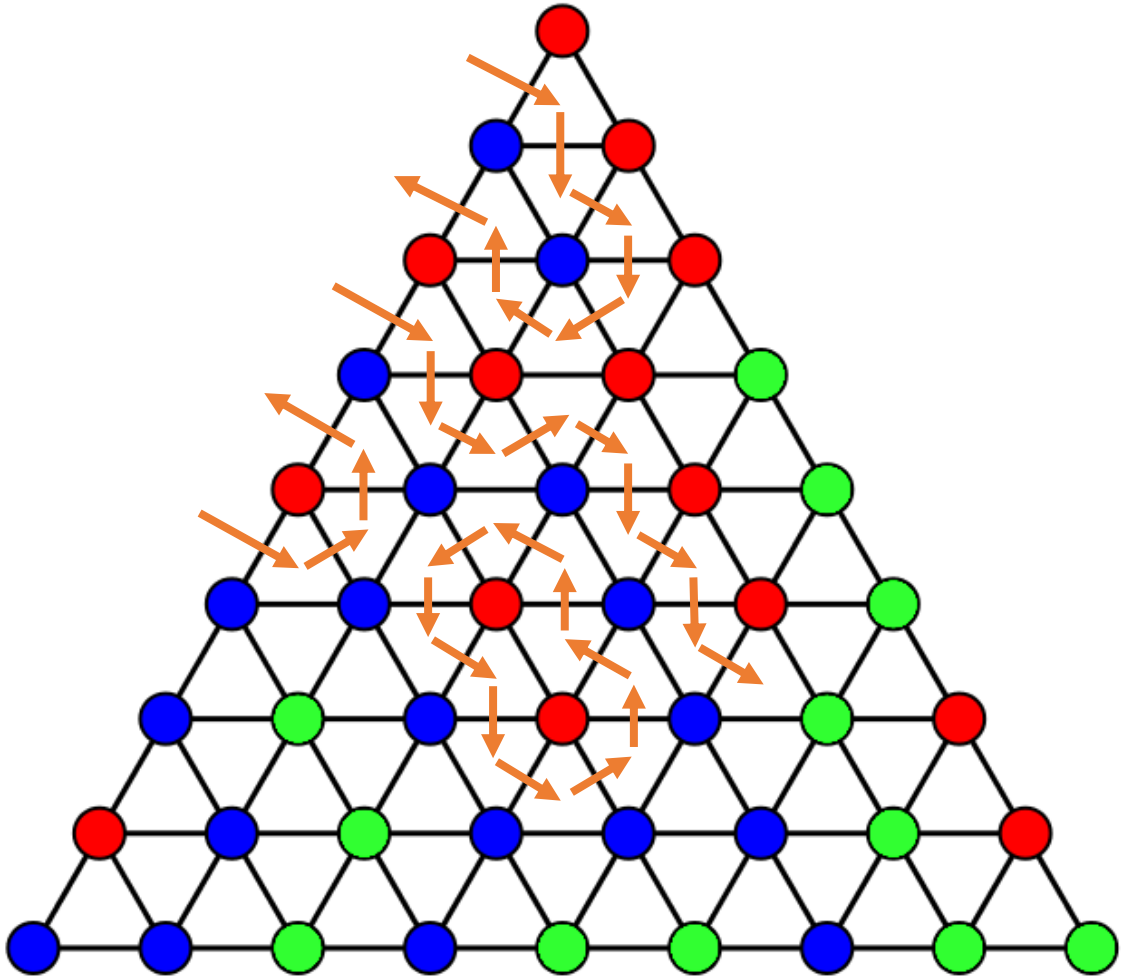

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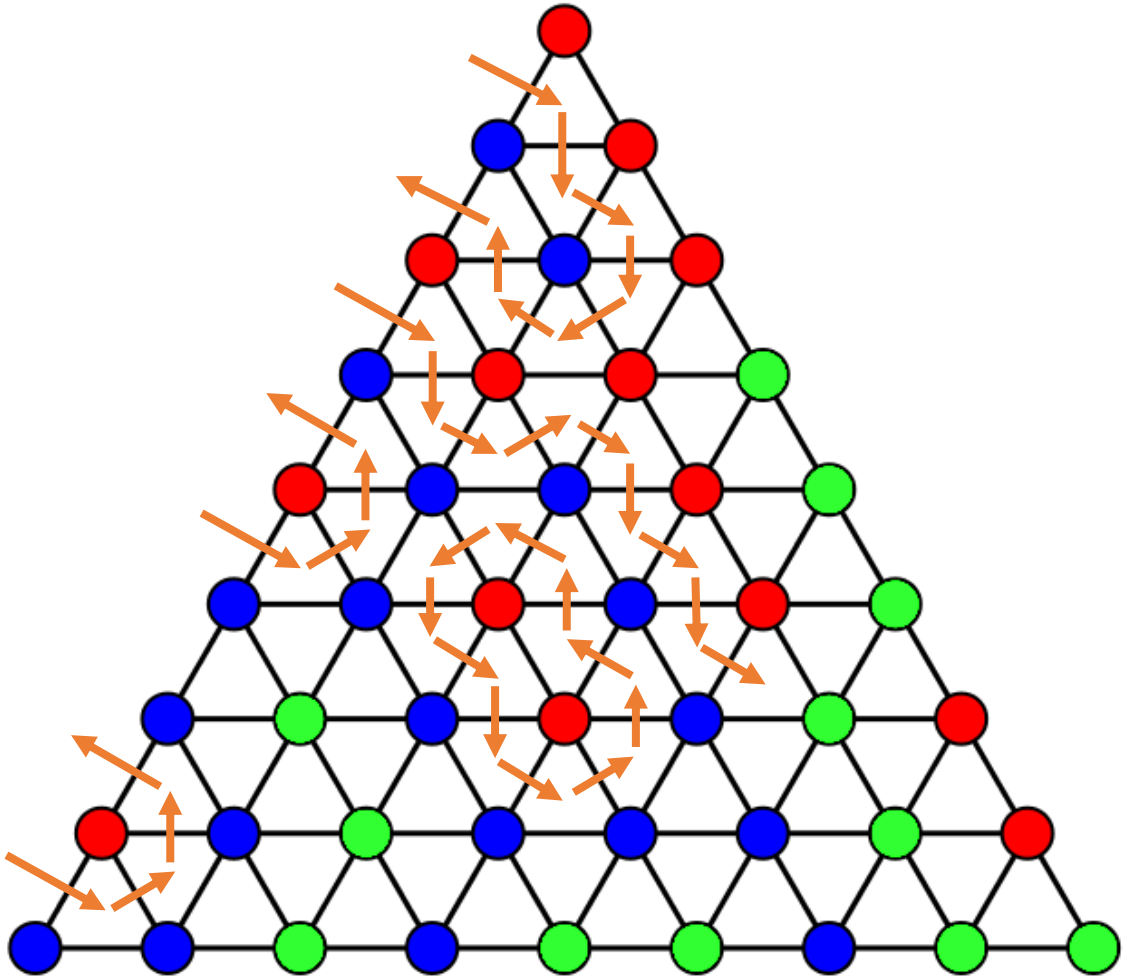

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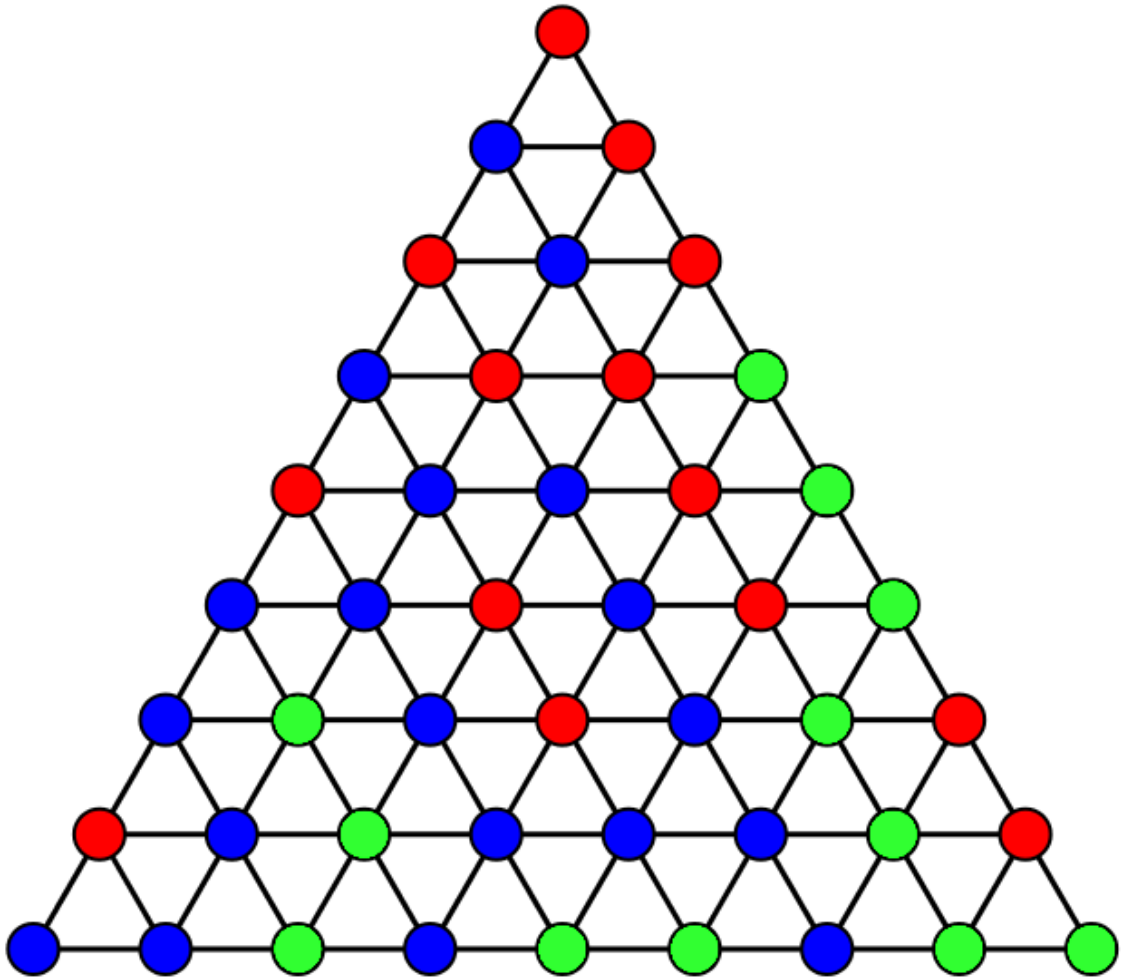

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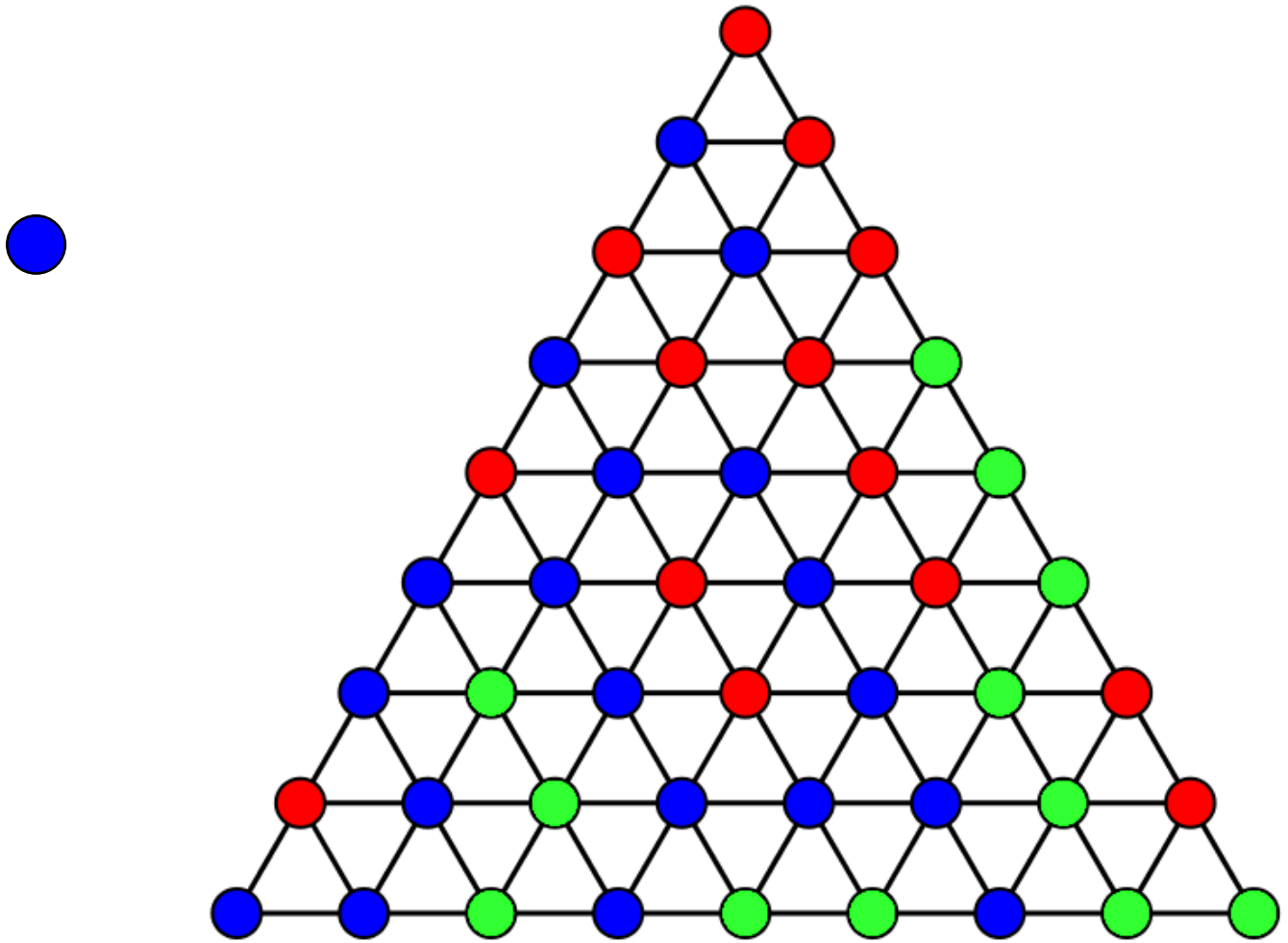

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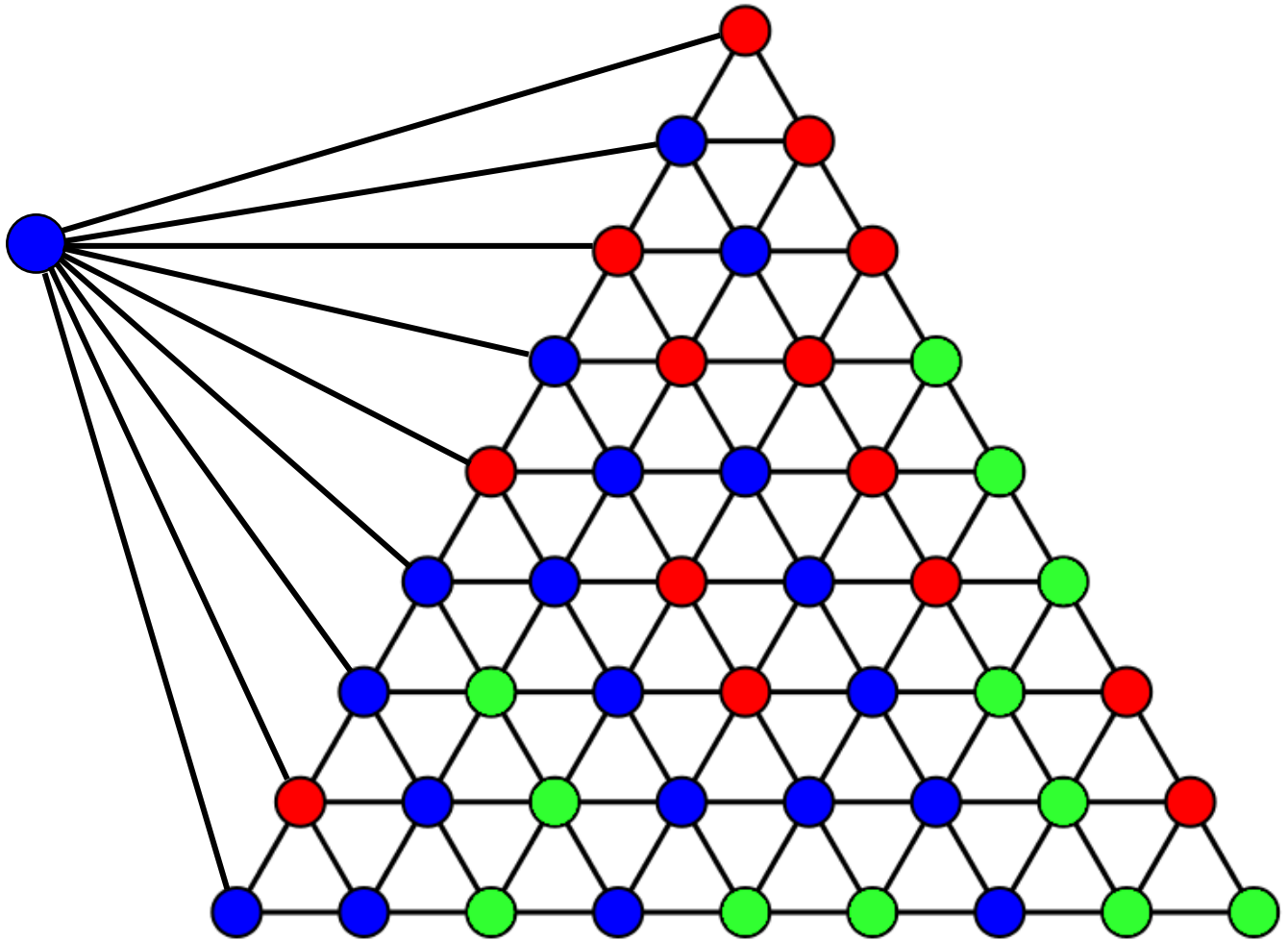

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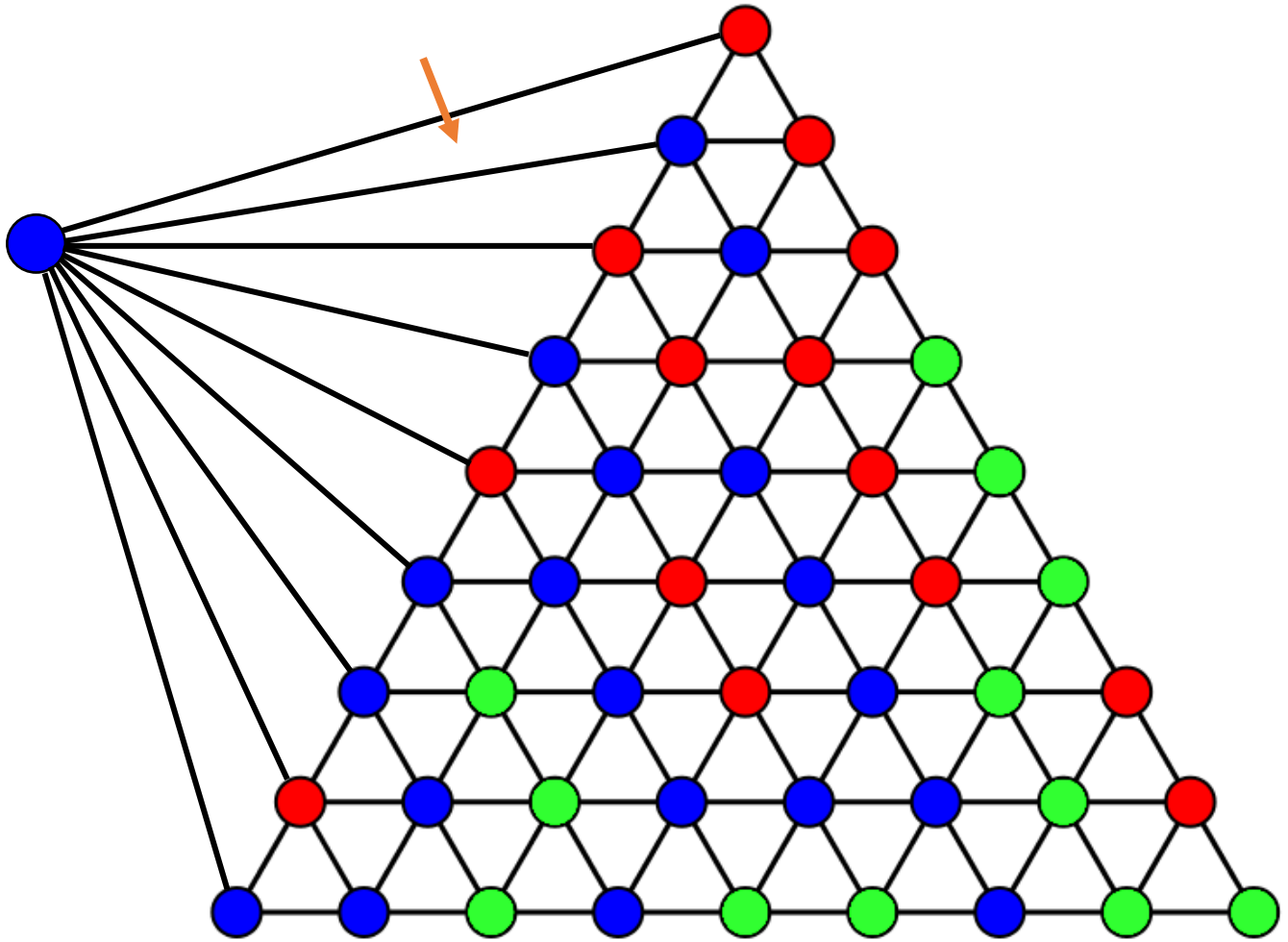

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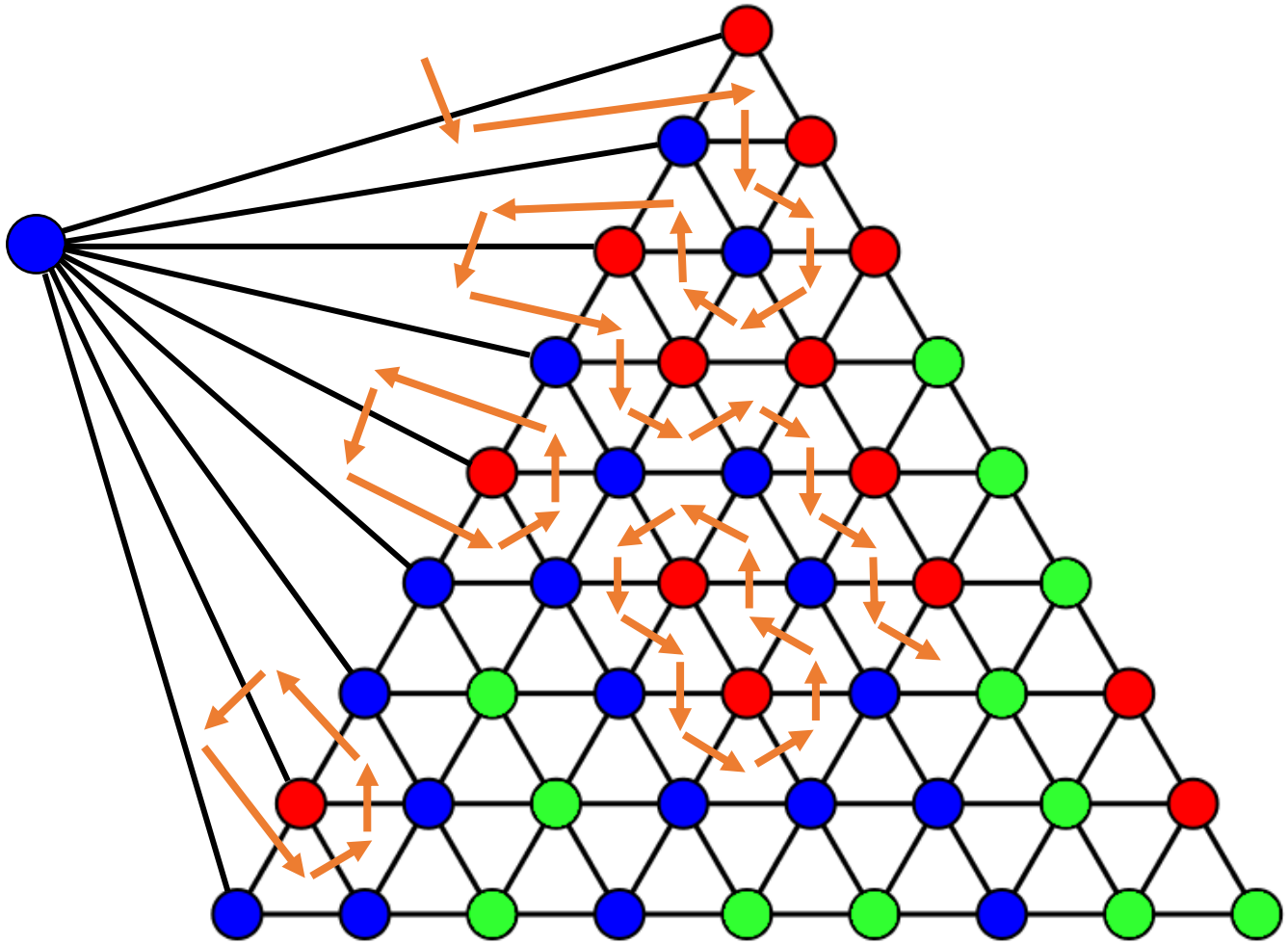

Focus on PPAD

I. How do we prove PPAD-membership?
   1. Sperner's Lemma
   2. **The END-OF-LINE Problem**

II. How do we prove PPAD-hardness?

Formal Definition of PPAD

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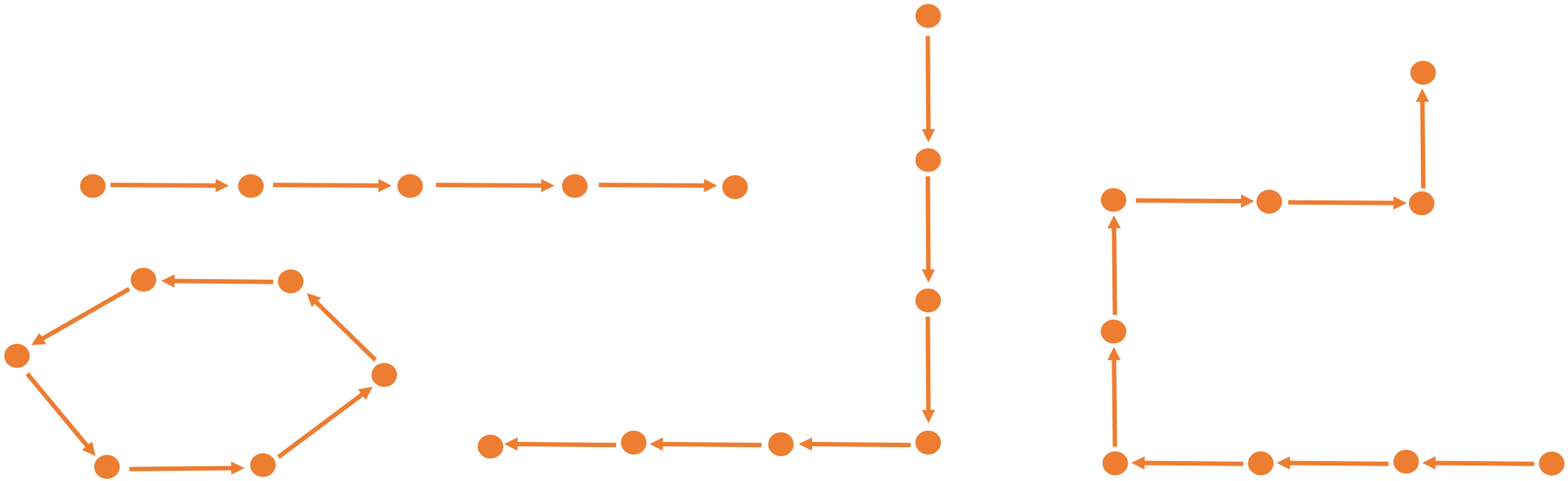

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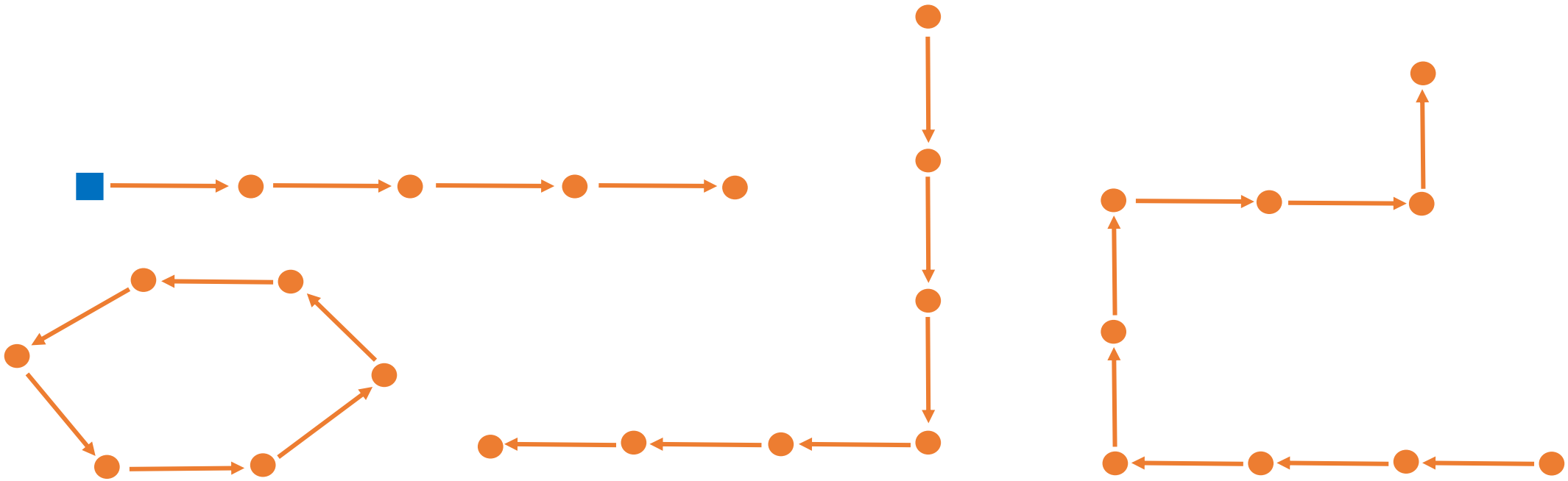

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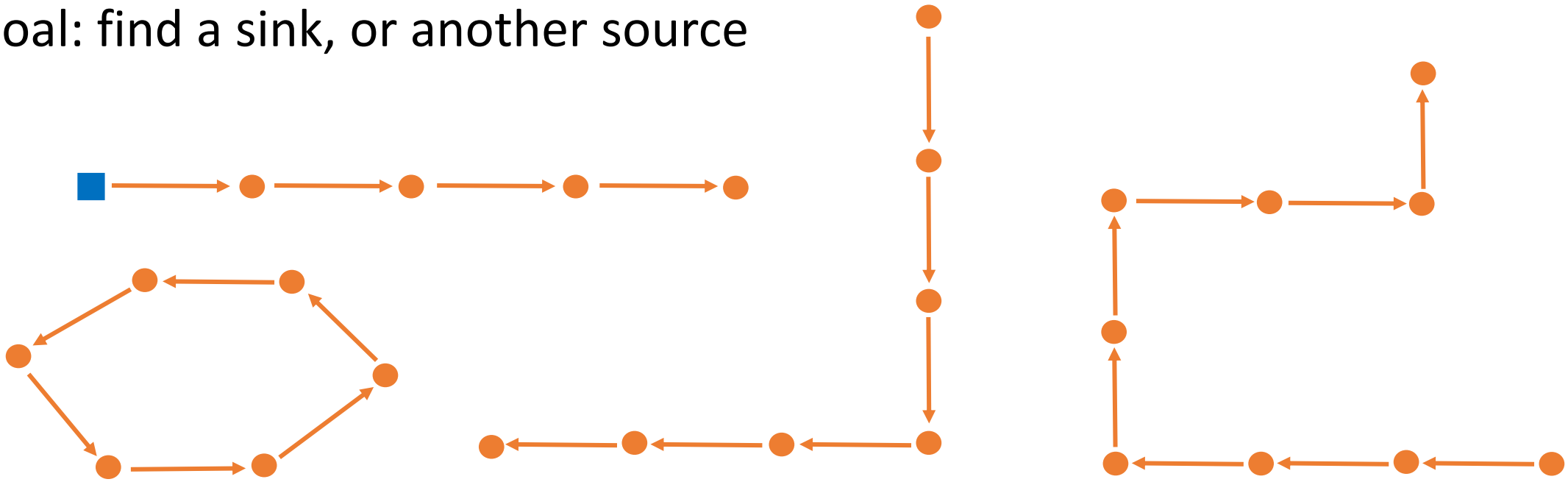
- Input: directed graph of paths and cycles, and a source
- Goal: find a sink, or another source



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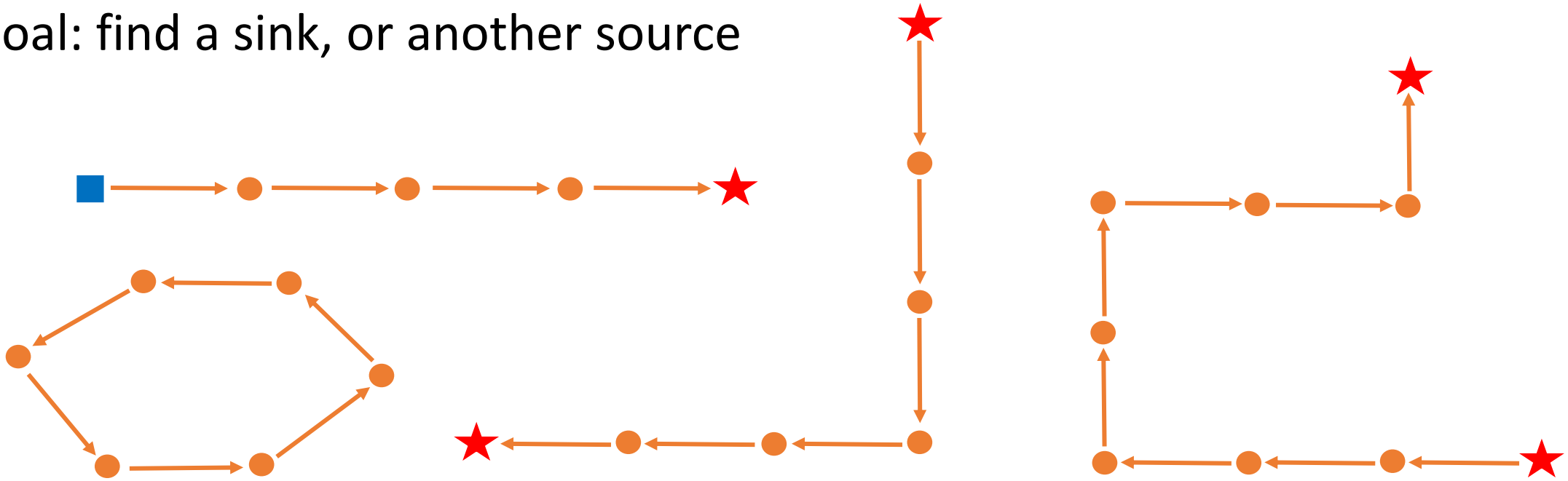

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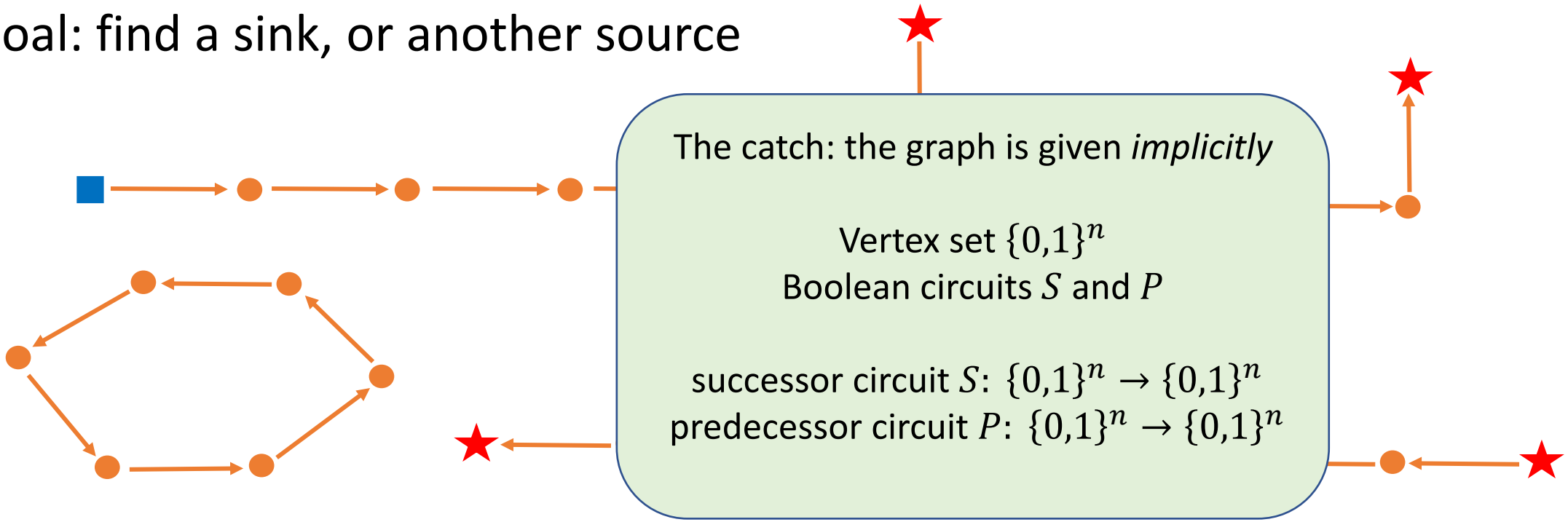
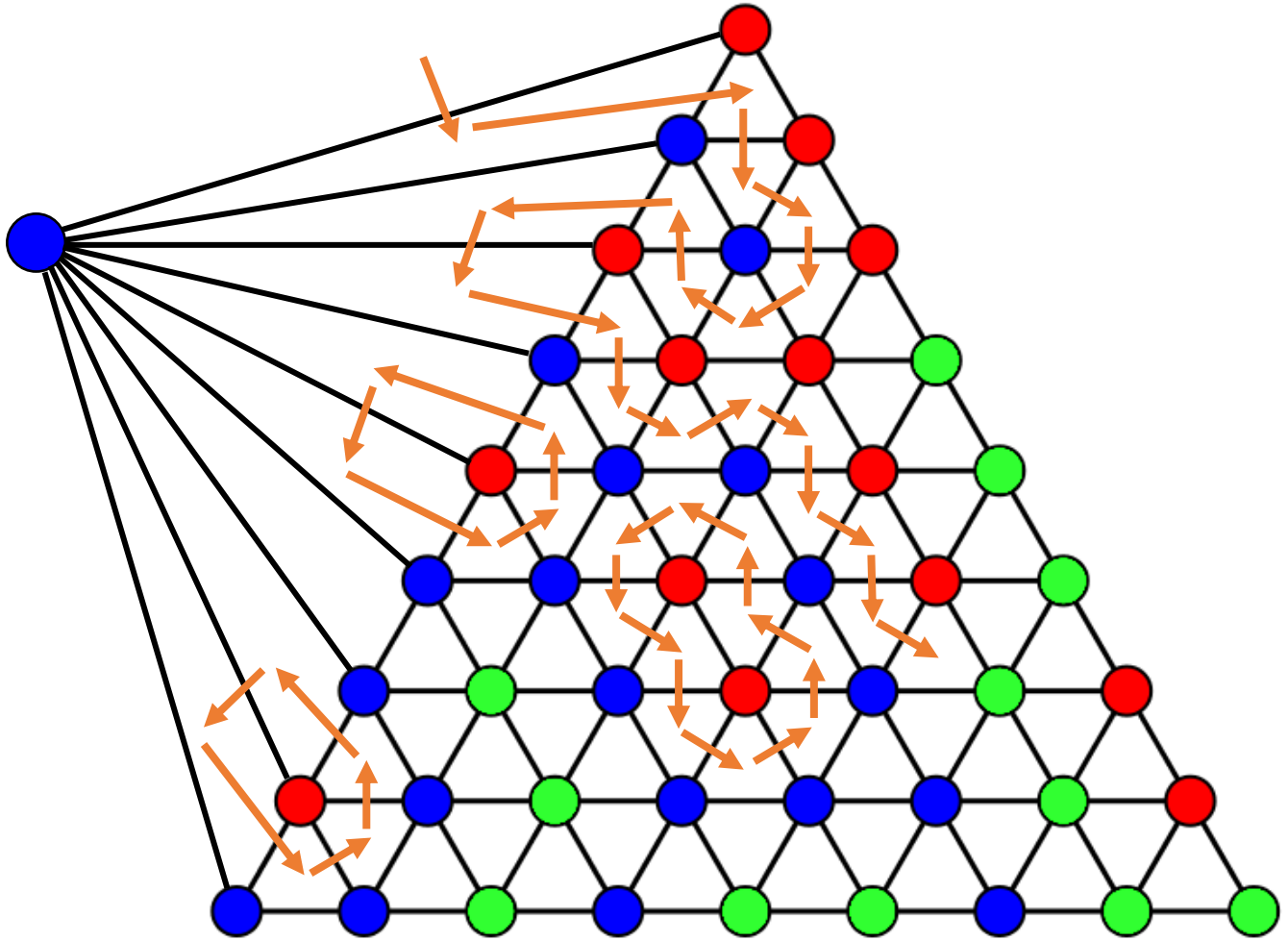


SPERNER lies in PPAD



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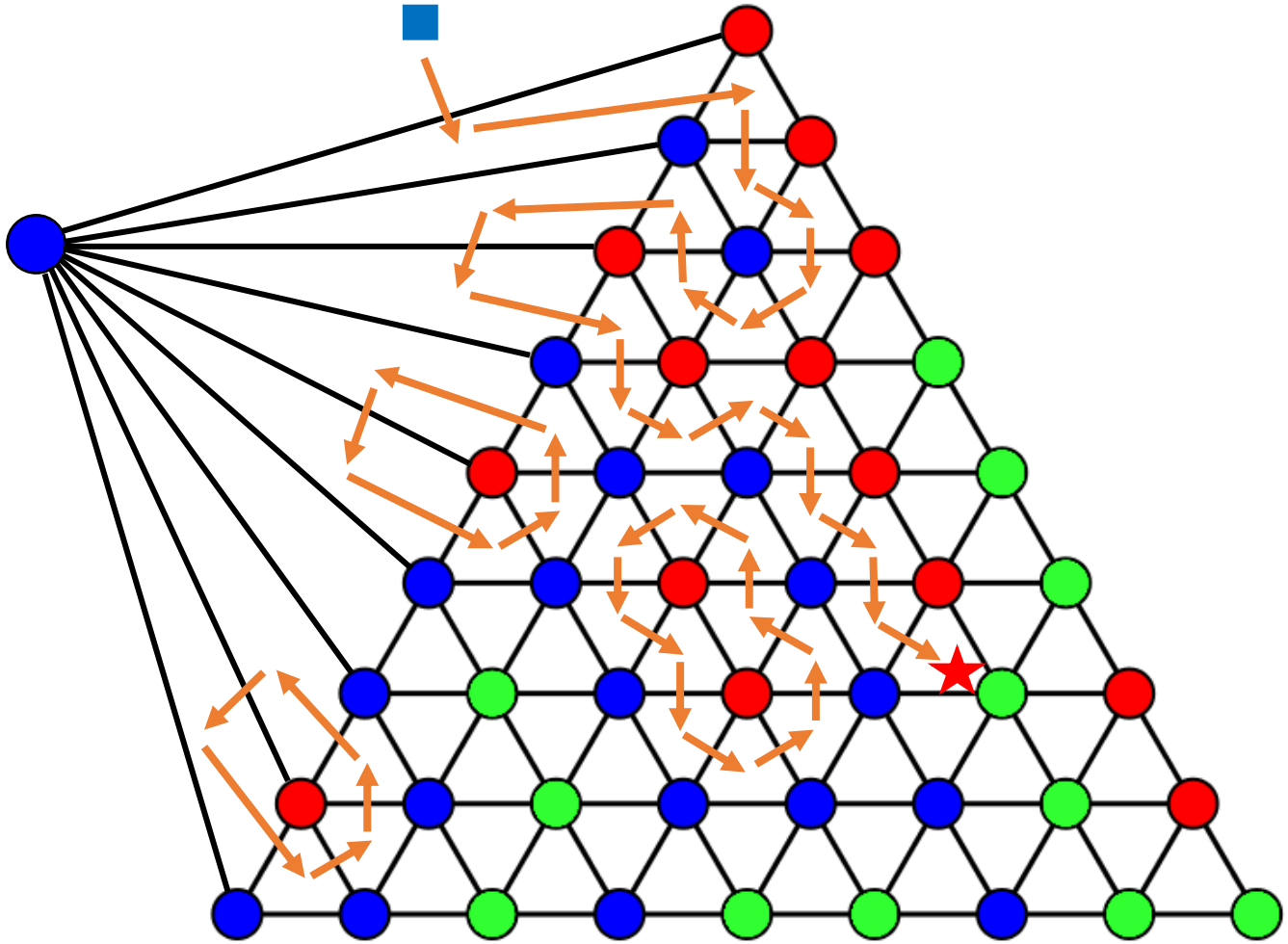

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   1. Sperner's Lemma
   2. The END-OF-LINE Problem
   3. **Example: NASH**

II. How do we prove PPAD-hardness?

The NASH Problem

A set of players with actions (pure strategies) $a_i, i = 1, \dots, n$. Given a vector of strategies $a = (a_1, a_2, \dots, a_n)$ (strategy profile), every agent receives a payoff $u_i(a)$.

Agents choose mixed strategies, i.e., probability mixtures over actions.

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Theorem (Nash 1951): Every finite strategic game has a Nash equilibrium (in mixed strategies)

Proof of Nash’s Theorem

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Proof of Nash's Theorem:

Let $D = \Delta_{m_1} \times \cdots \times \Delta_{m_n}$. Consider $F: D \to D$ defined by $F_{i,j}(x) := \frac{x_{i,j} + g_{i,j}(x)}{1 + \sum_{k=1}^{m_i} g_{i,k}(x)}$

where $g_{i,j}(x) := \max\{0,\ u_i(a_{i,j}, x_{-i}) - u_i(x)\}$.

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→ any fixed point x of F is a Nash equilibrium!

PPAD-membership of NASH

Replace Brouwer by Sperner:

PPAD-membership of NASH

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Brouwer function $F: D \to D$

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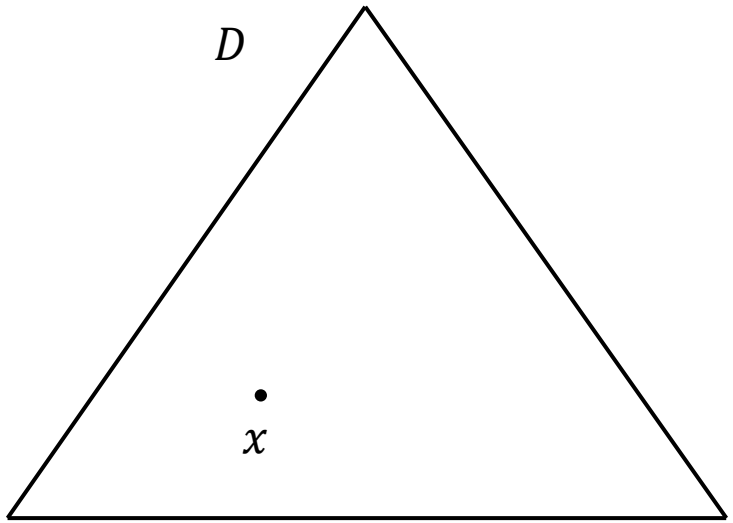

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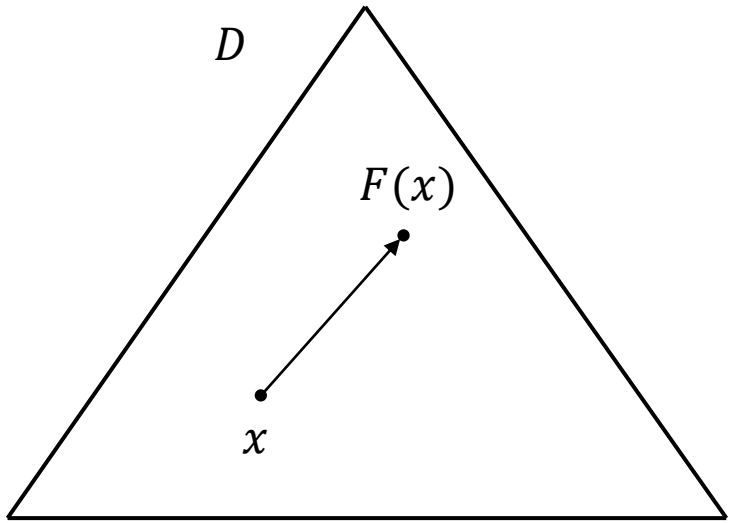

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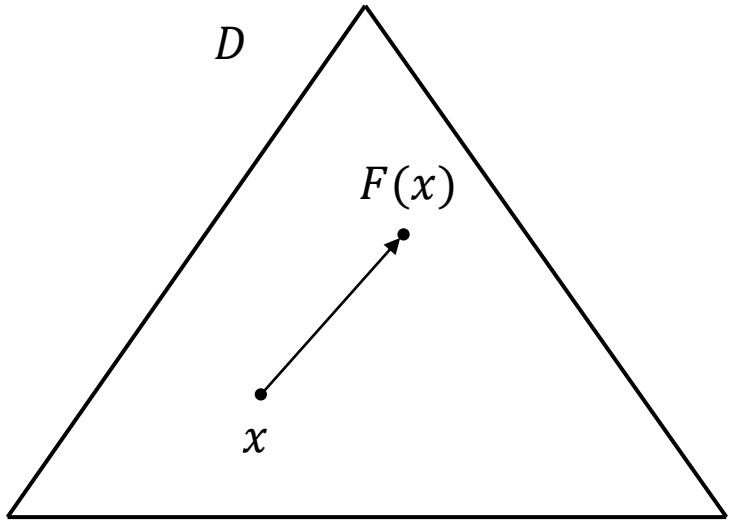

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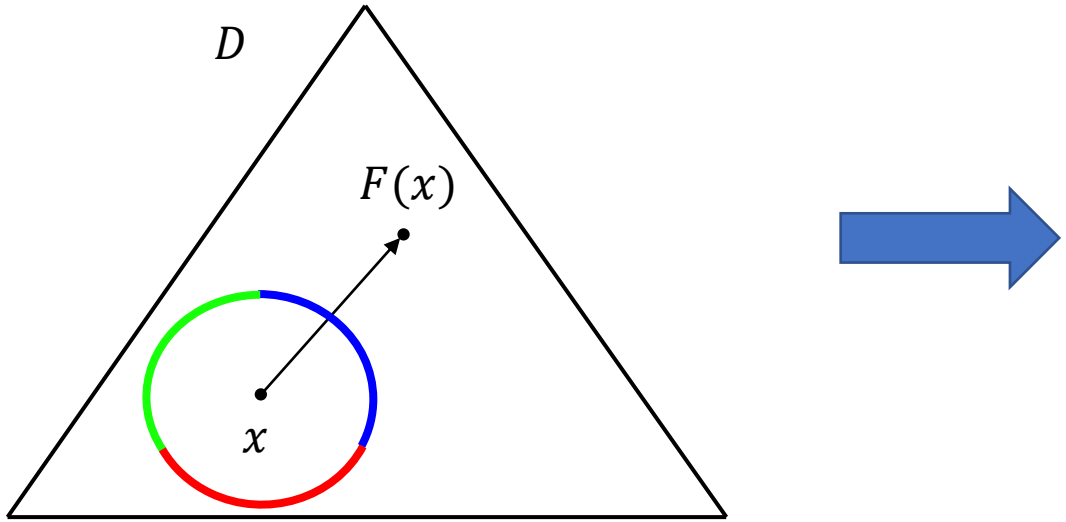

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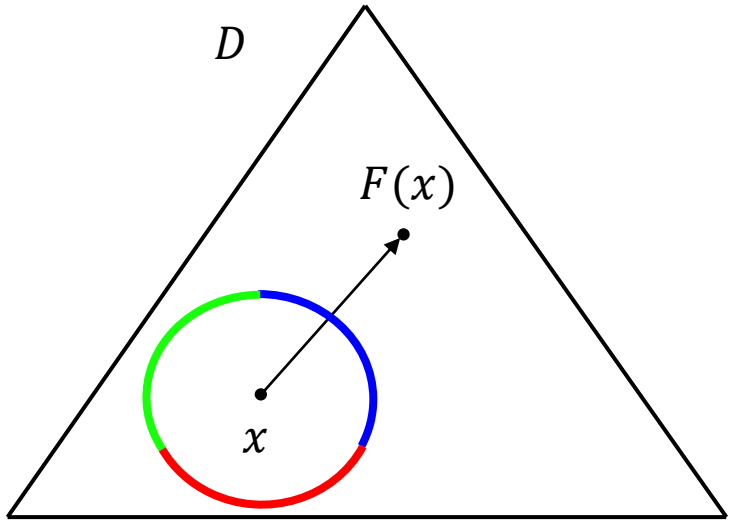

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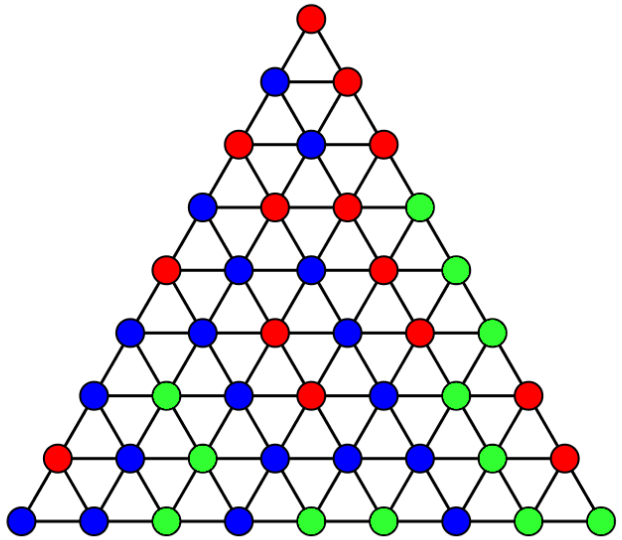

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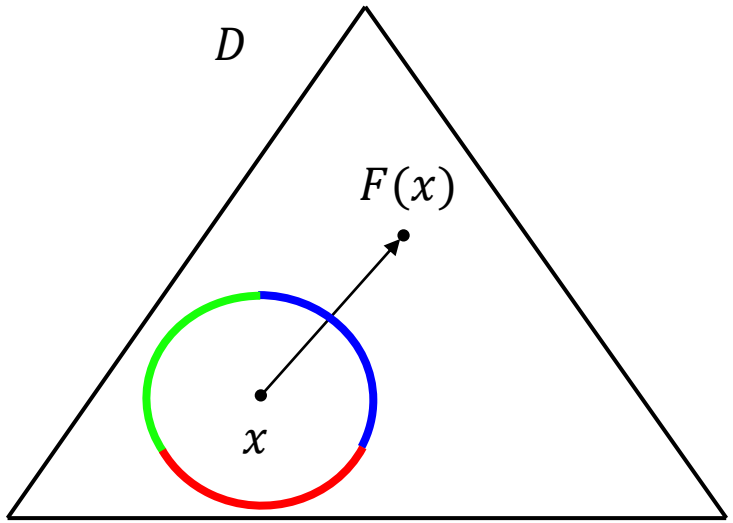

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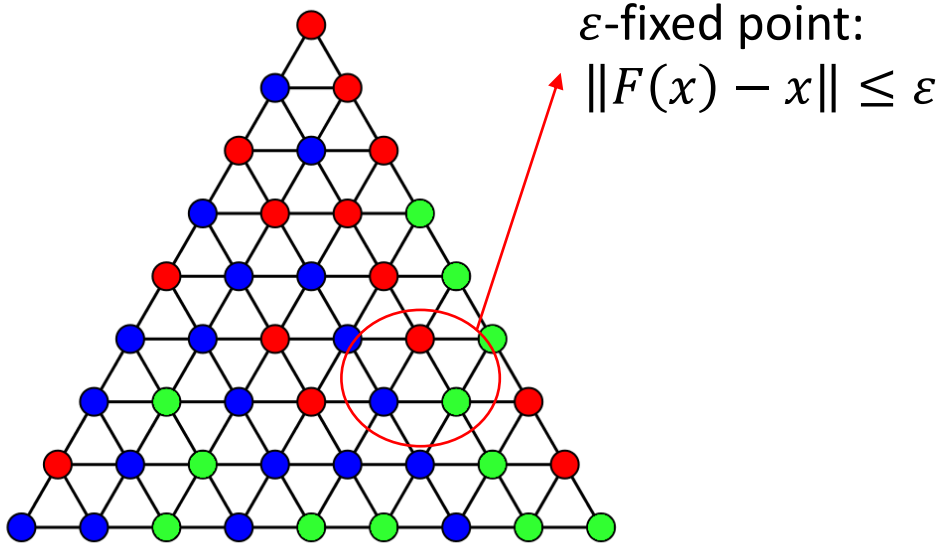

Focus on PPAD

I. How do we prove PPAD-membership?
   1. Sperner's Lemma
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   3. Example: NASH

II. How do we prove PPAD-hardness?
   1. **Refresher: Reductions**

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Intuition: “if we can solve A, then we can solve 3-SAT”

How to prove PPAD-hardness?

Refresher: How to prove PPAD-hardness of NASH?

→ reduce from an PPAD-complete problem, e.g., END-OF-LINE

END-OF-LINE $\leq_p$ **NASH** : there exists a poly-time function f that maps instances of END-OF-LINE to instances of NASH, and such that for any instance x of END-OF-LINE:

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~~x is YES-instance for END-OF-LINE iff $f(x)$ is YES-instance for NASH~~

Given any solution y to instance $f(x)$ of NASH, we can efficiently obtain a solution to instance x of END-OF-LINE

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All subsequent work reduces from GENERALIZED-CIRCUIT or some version of NASH

Generalized Circuit

DEFINITION 3.4. *Given a generalized circuit $\mathcal{S} = (V, \mathcal{T})$, we say that an assignment* $\mathbf{x}\colon V \to [0,1]$ ϵ-approximately satisfies $\mathcal{S}$ *if for each of the following gates,* $\mathbf{x}$ *satisfies the corresponding constraints:*

Gate	*Constraint*
$G_\zeta\,(\alpha \mid\mid a)$	$\mathbf{x}[a] = \alpha \pm \epsilon$
$G_{\times\zeta}\,(\alpha \mid a \mid b)$	$\mathbf{x}[b] = \alpha \cdot \mathbf{x}[a] \pm \epsilon$
$G_=\,(\mid a \mid b)$	$\mathbf{x}[b] = \mathbf{x}[a] \pm \epsilon$
$G_+\,(\mid a, b \mid c)$	$\mathbf{x}[c] = \min(\mathbf{x}[a] + \mathbf{x}[b], 1) \pm \epsilon$
$G_-\,(\mid a, b \mid c)$	$\mathbf{x}[c] = \max(\mathbf{x}[a] - \mathbf{x}[b], 0) \pm \epsilon$
$G_<\,(\mid a, b \mid c)$	$\mathbf{x}[c] = \begin{cases} 1 \pm \epsilon & \mathbf{x}[a] < \mathbf{x}[b] - \epsilon \\ 0 \pm \epsilon & \mathbf{x}[a] > \mathbf{x}[b] + \epsilon \end{cases}$
$G_\vee\,(\mid a, b \mid c)$	$\mathbf{x}[c] = \begin{cases} 1 \pm \epsilon & \mathbf{x}[a] = 1 \pm \epsilon \text{ or } \mathbf{x}[b] = 1 \pm \epsilon \\ 0 \pm \epsilon & \mathbf{x}[a] = 0 \pm \epsilon \text{ and } \mathbf{x}[b] = 0 \pm \epsilon \end{cases}$
$G_\wedge\,(\mid a, b \mid c)$	$\mathbf{x}[c] = \begin{cases} 1 \pm \epsilon & \mathbf{x}[a] = 1 \pm \epsilon \text{ and } \mathbf{x}[b] = 1 \pm \epsilon \\ 0 \pm \epsilon & \mathbf{x}[a] = 0 \pm \epsilon \text{ or } \mathbf{x}[b] = 0 \pm \epsilon \end{cases}$
$G_\neg\,(\mid a \mid b)$	$\mathbf{x}[b] = \begin{cases} 1 \pm \epsilon & \mathbf{x}[a] = 0 \pm \epsilon \\ 0 \pm \epsilon & \mathbf{x}[a] = 1 \pm \epsilon \end{cases}$

(where G_ζ and $G_{\times\zeta}$ also take a parameter $\alpha \in [0,1]$).

*Given a generalized circuit $\mathcal{S} = (V, \mathcal{T})$, ϵ-*GCIRCUIT *is the problem of finding an assignment that ϵ-approximately satisfies it.*

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   2. **The PURE-CIRCUIT Problem**

The Pure-Circuit Problem

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Input: A Boolean circuit, with a twist:

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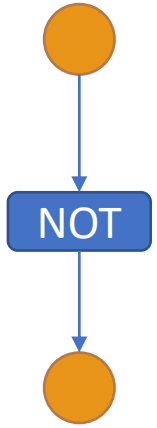
3. In addition to the standard logical gates (NOT, OR, AND), the circuit can also have “PURIFY” gates

Goal: Assign a value (in $\{0,1,\perp\}$) to each node, such that all gates are “satisfied”

The Pure-Circuit Problem

NOT gate:



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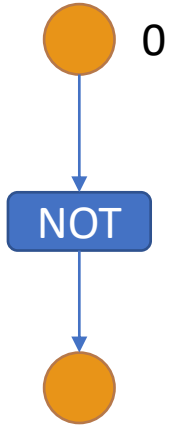

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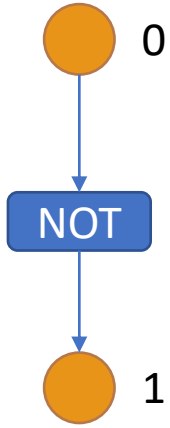

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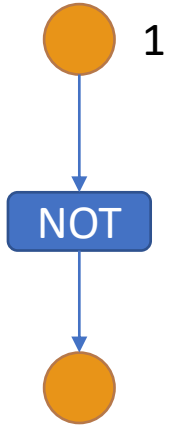

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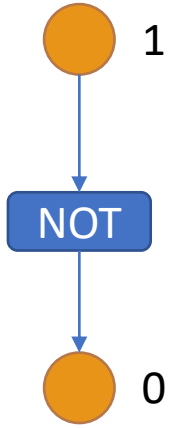

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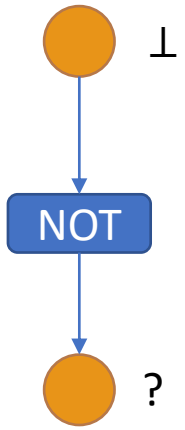

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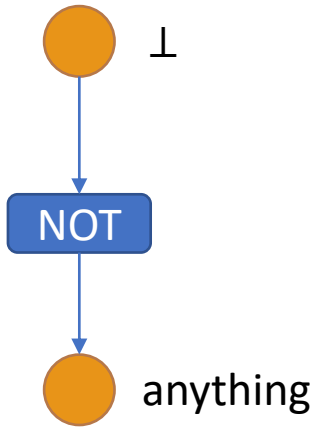

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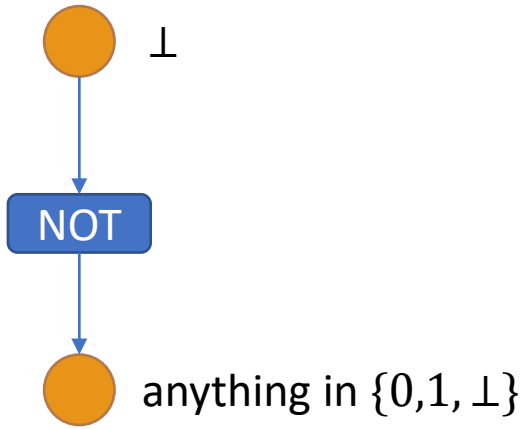

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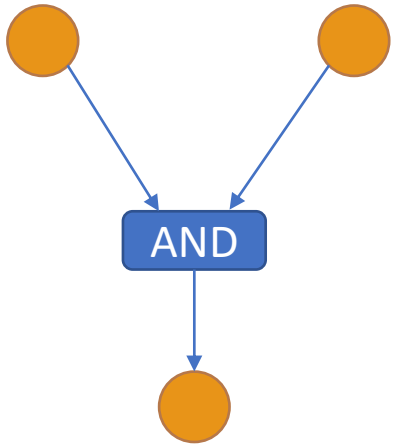
AND gate:



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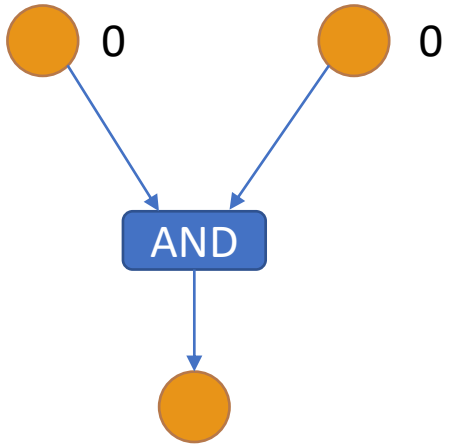

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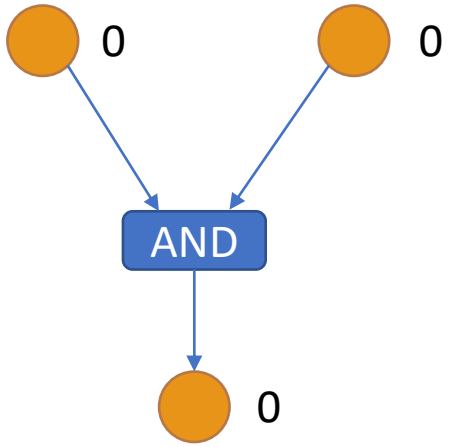

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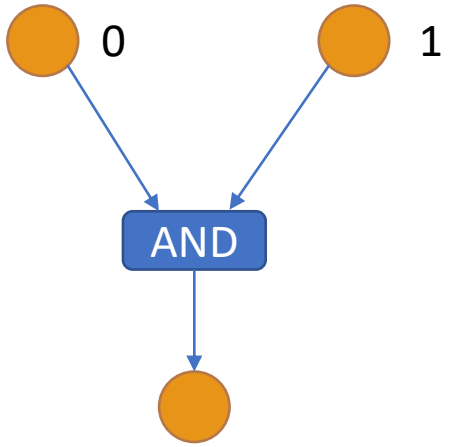

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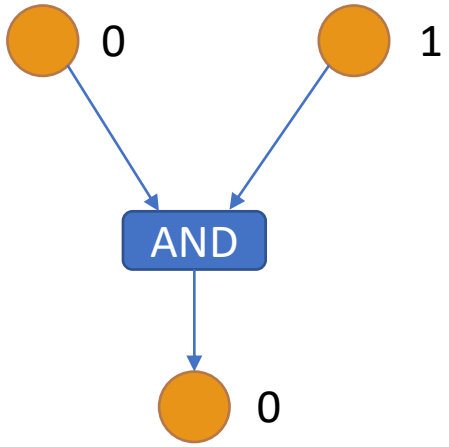

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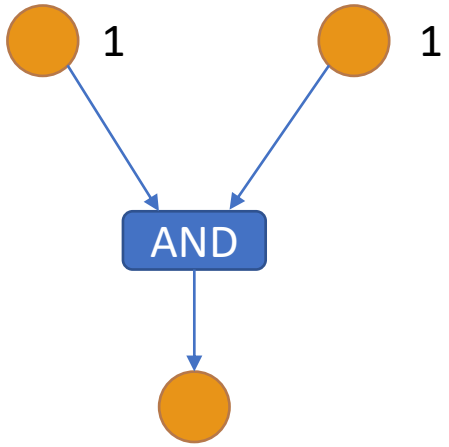

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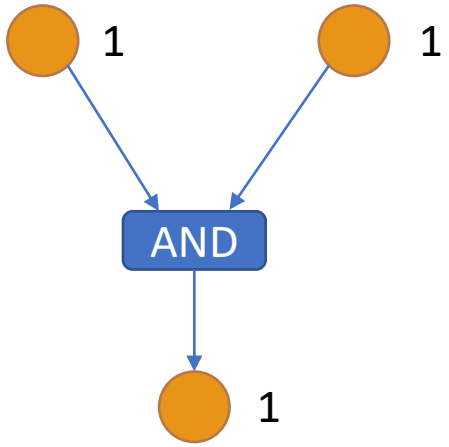

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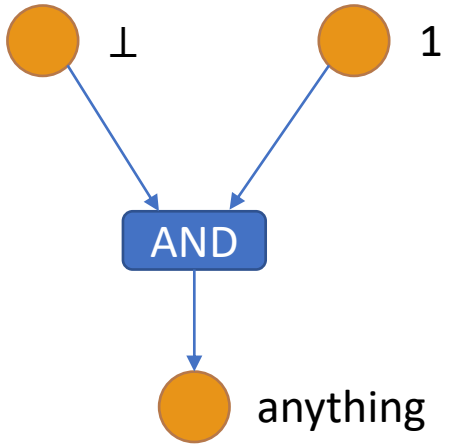

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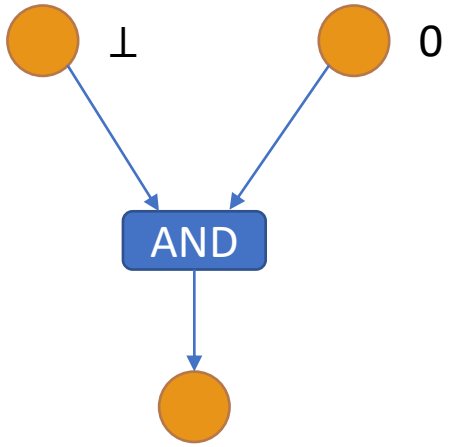

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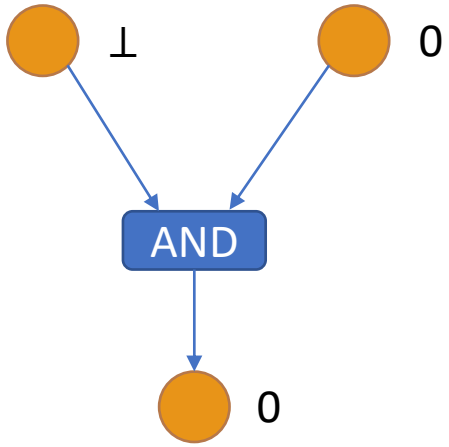
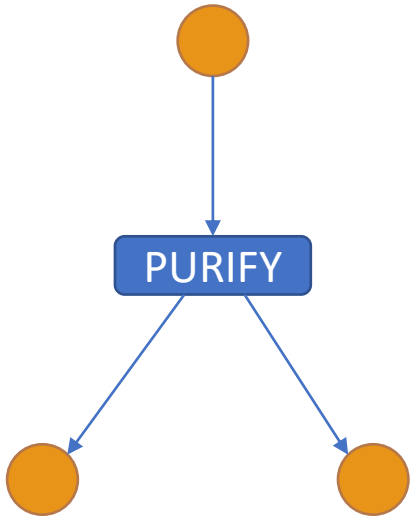


(with robustness!)

The Pure-Circuit Problem

PURIFY gate:



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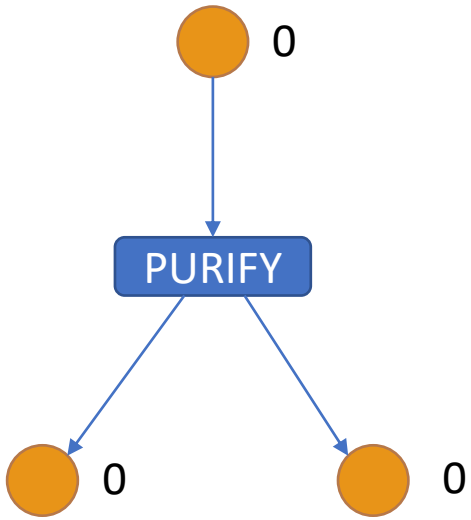

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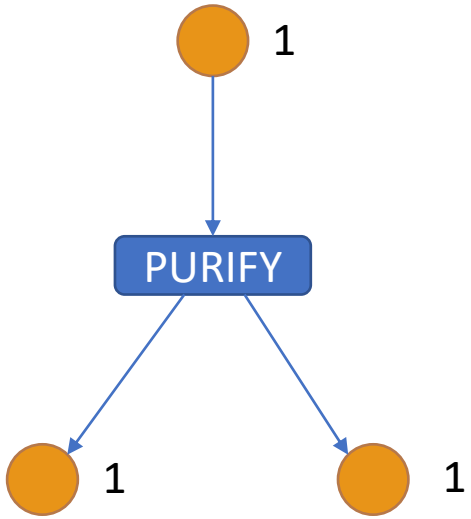

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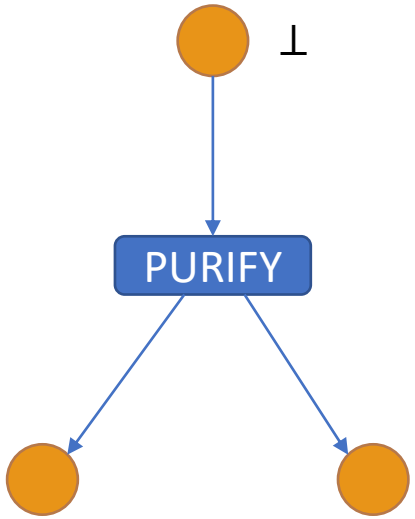

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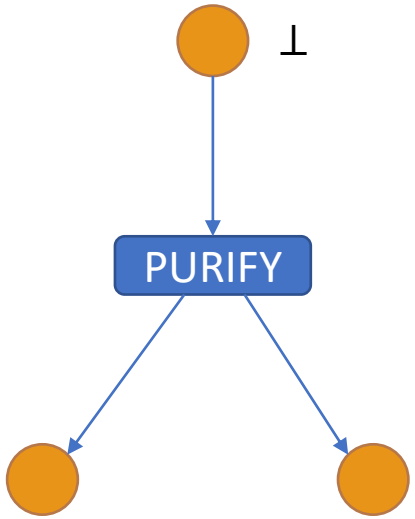

at least one in {0,1}

The Pure-Circuit Problem

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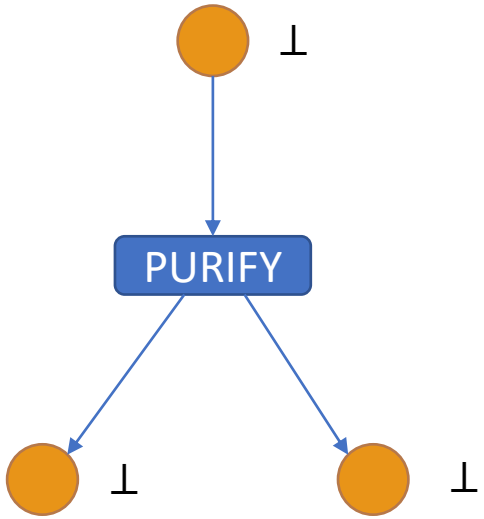

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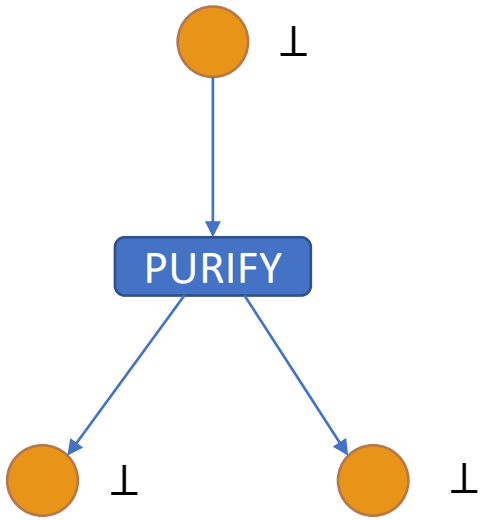

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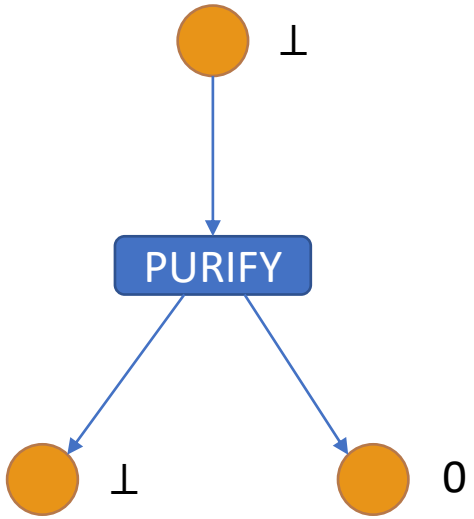

The Pure-Circuit Problem: Examples

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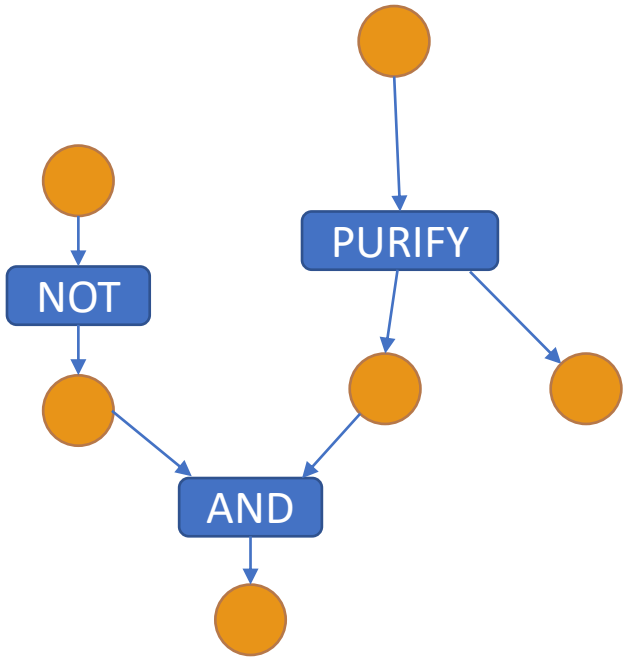

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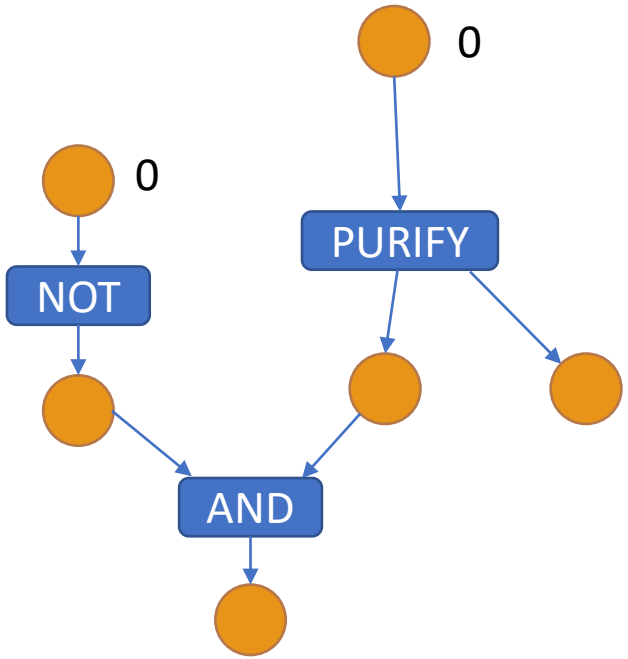

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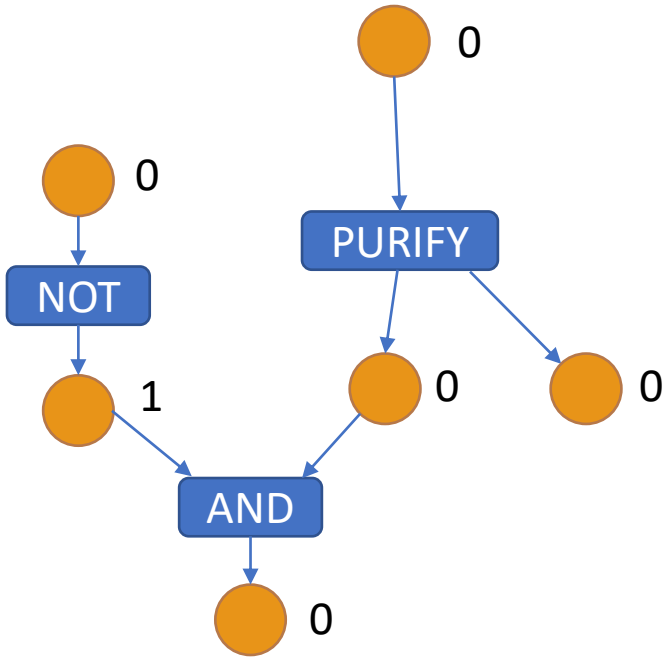

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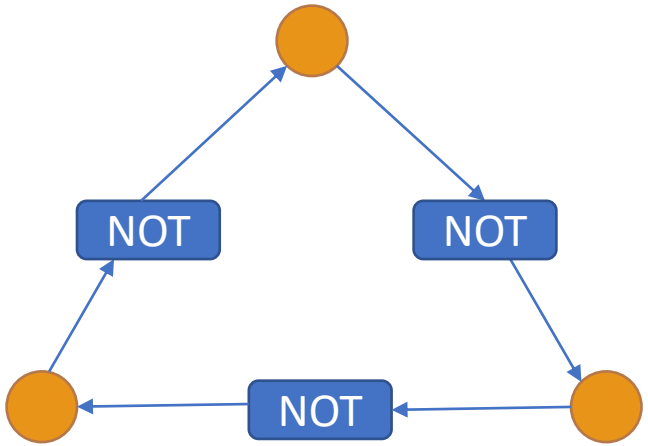

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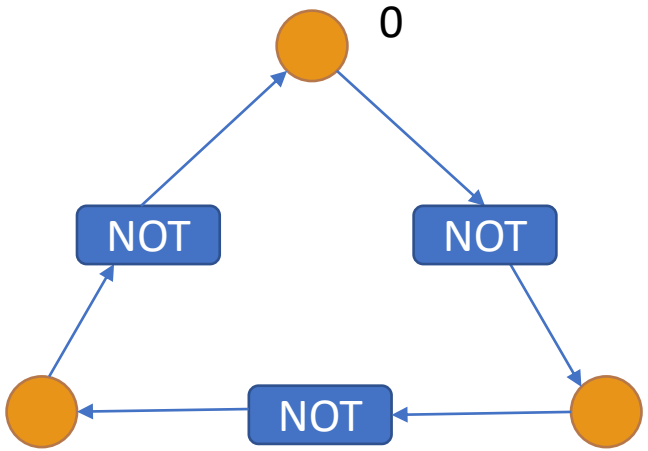

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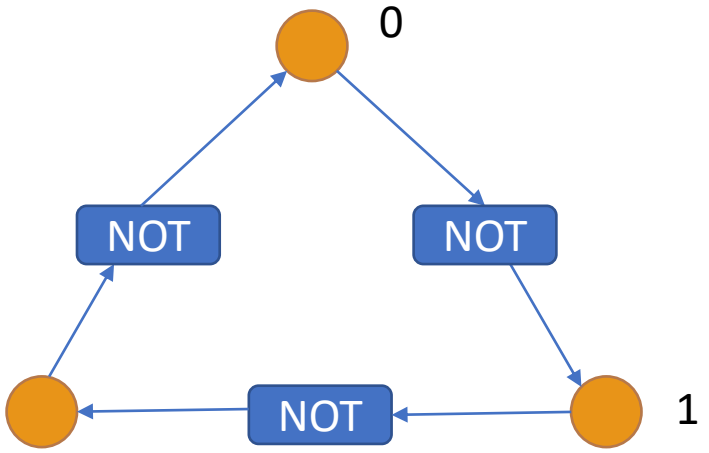

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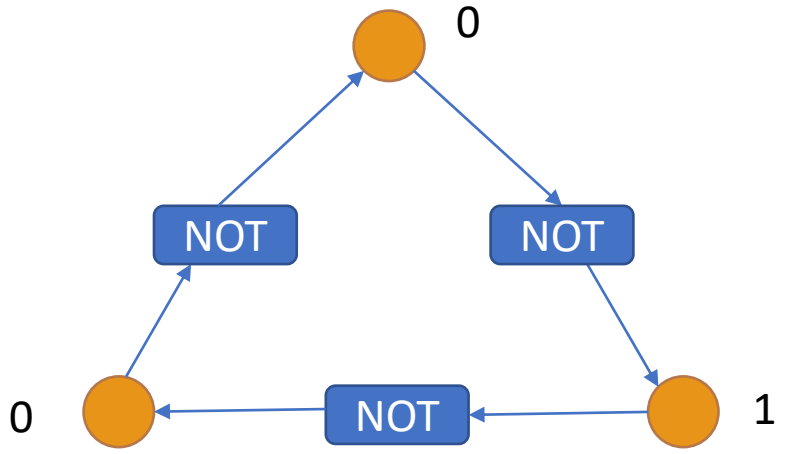

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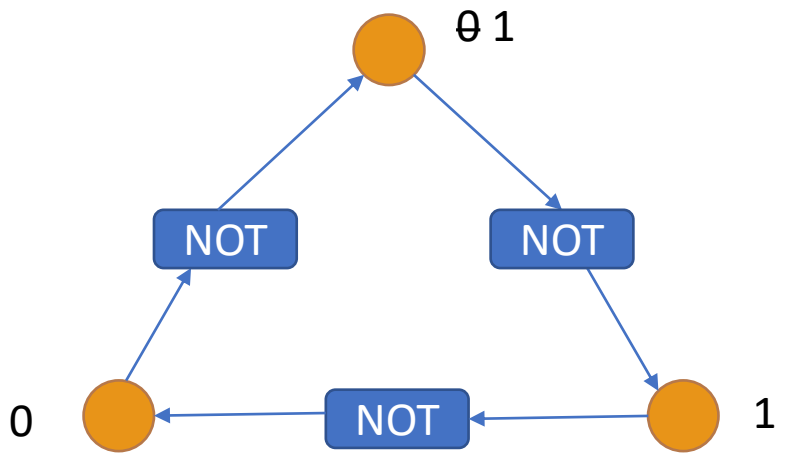

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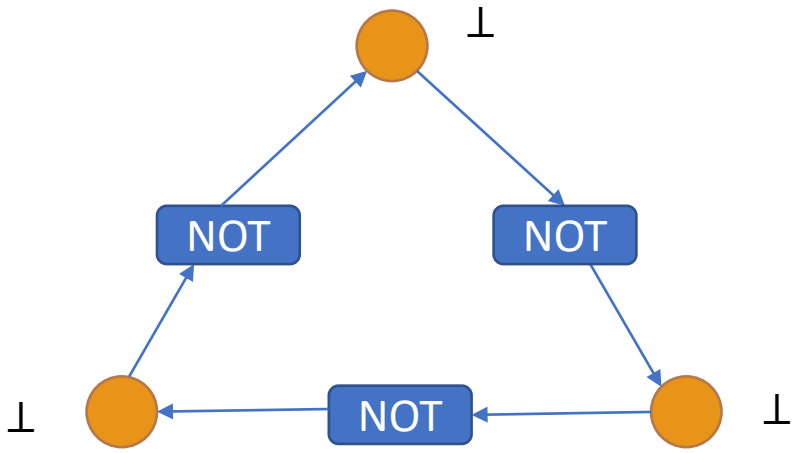

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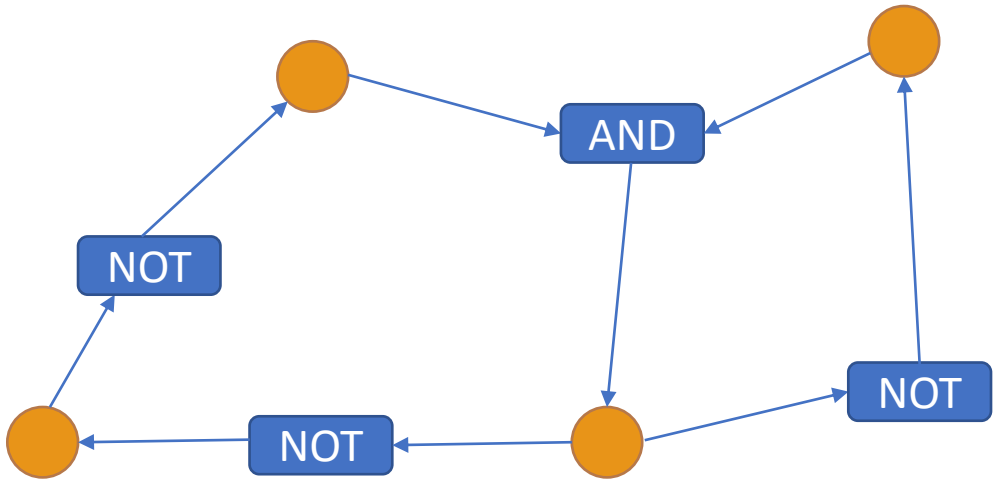

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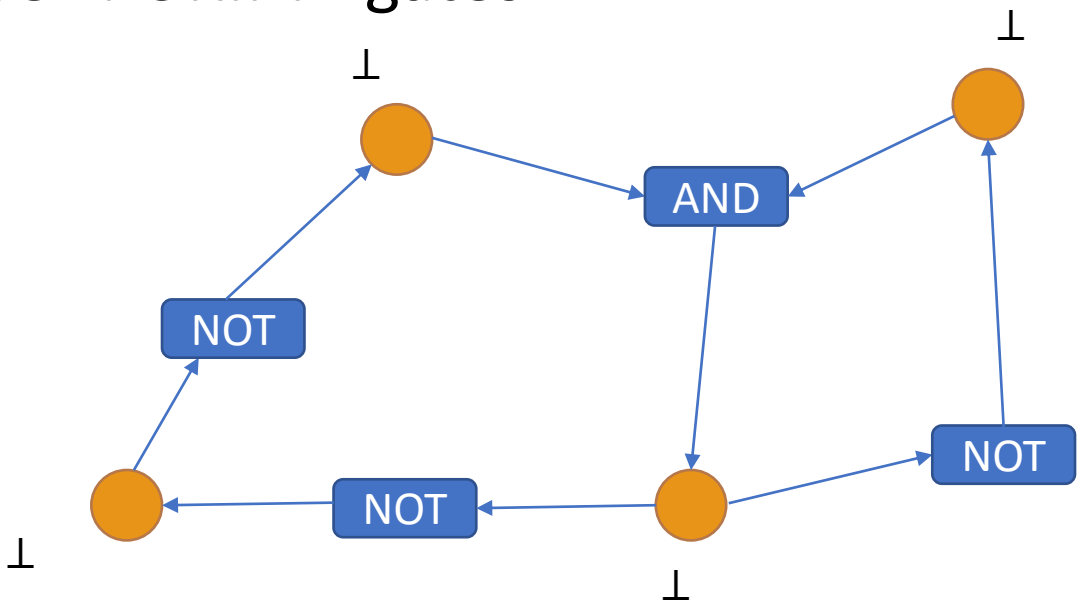

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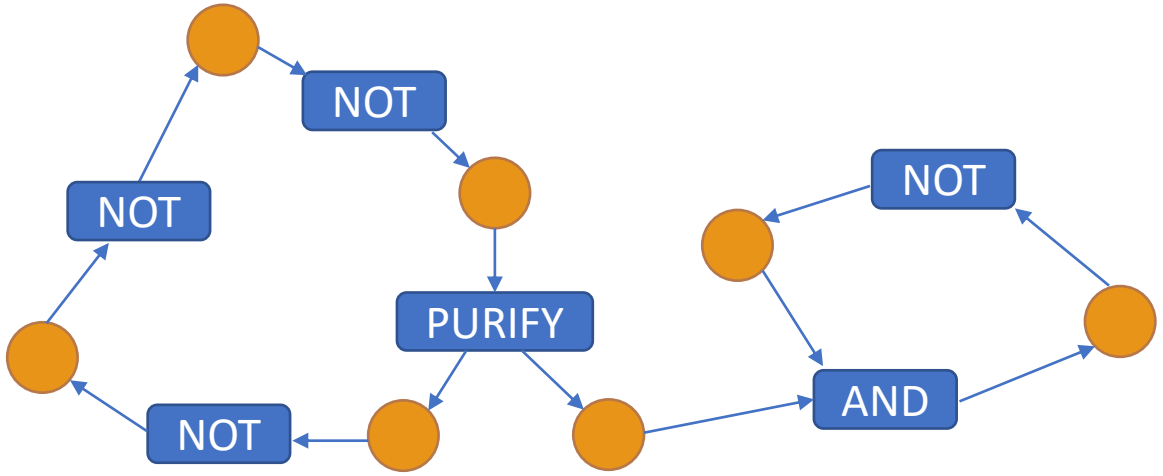

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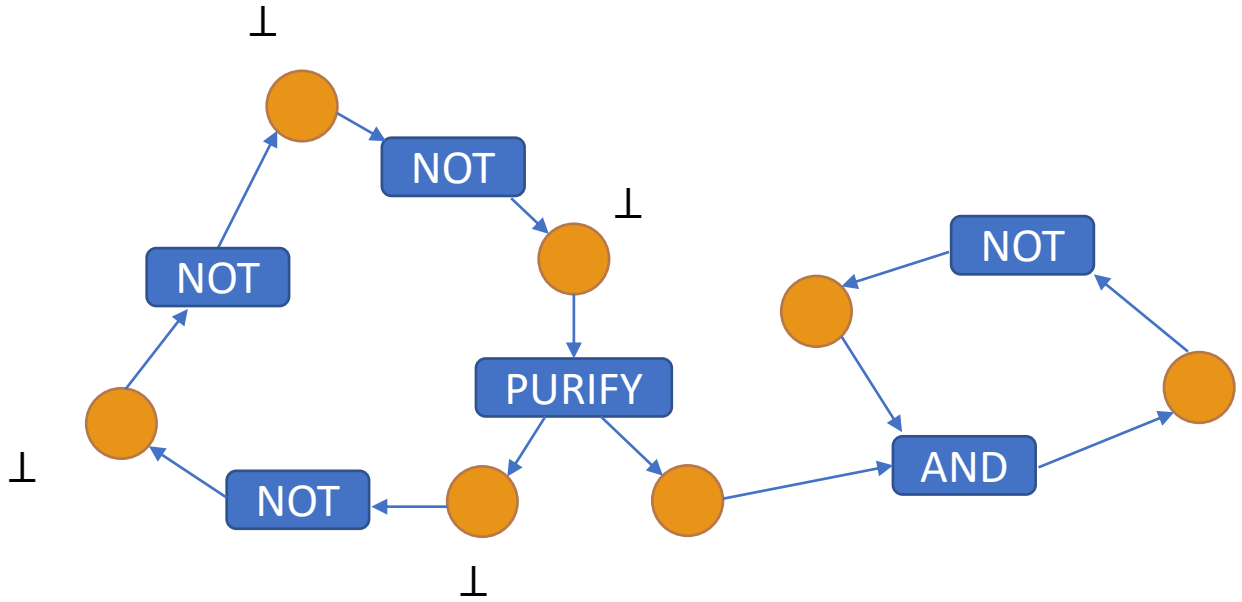

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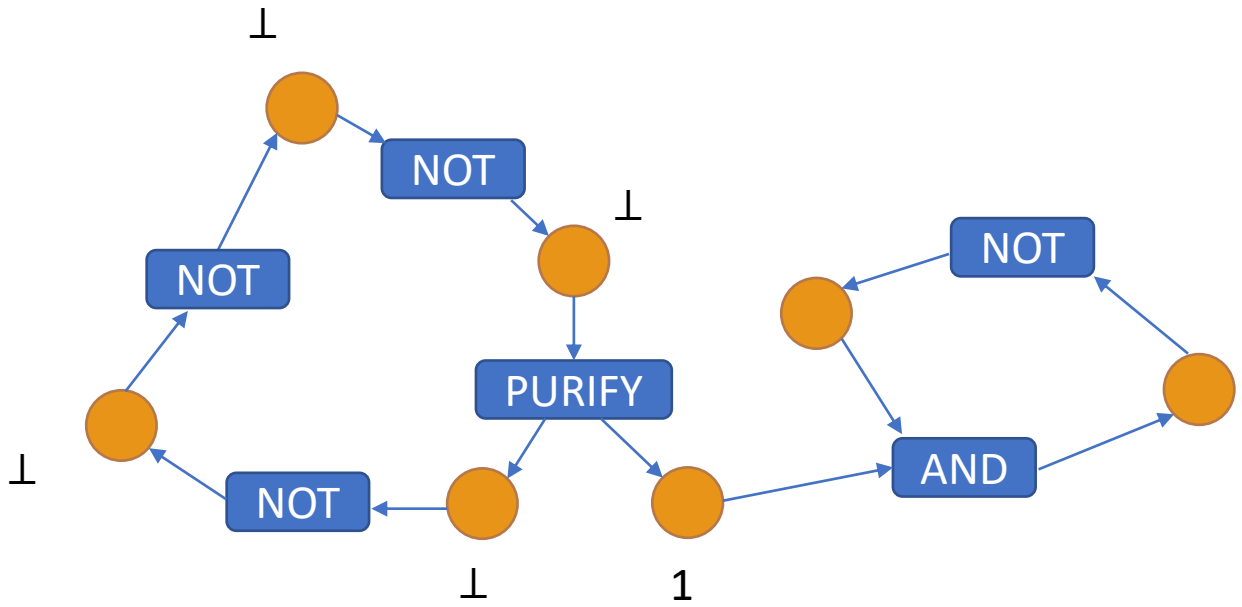

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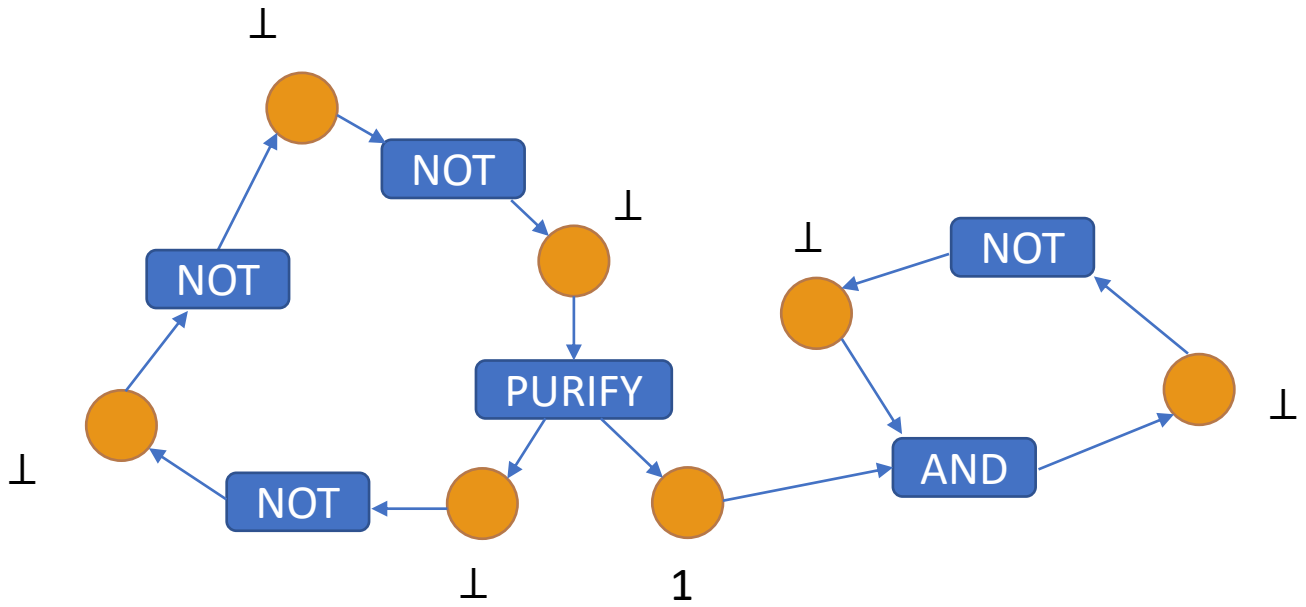

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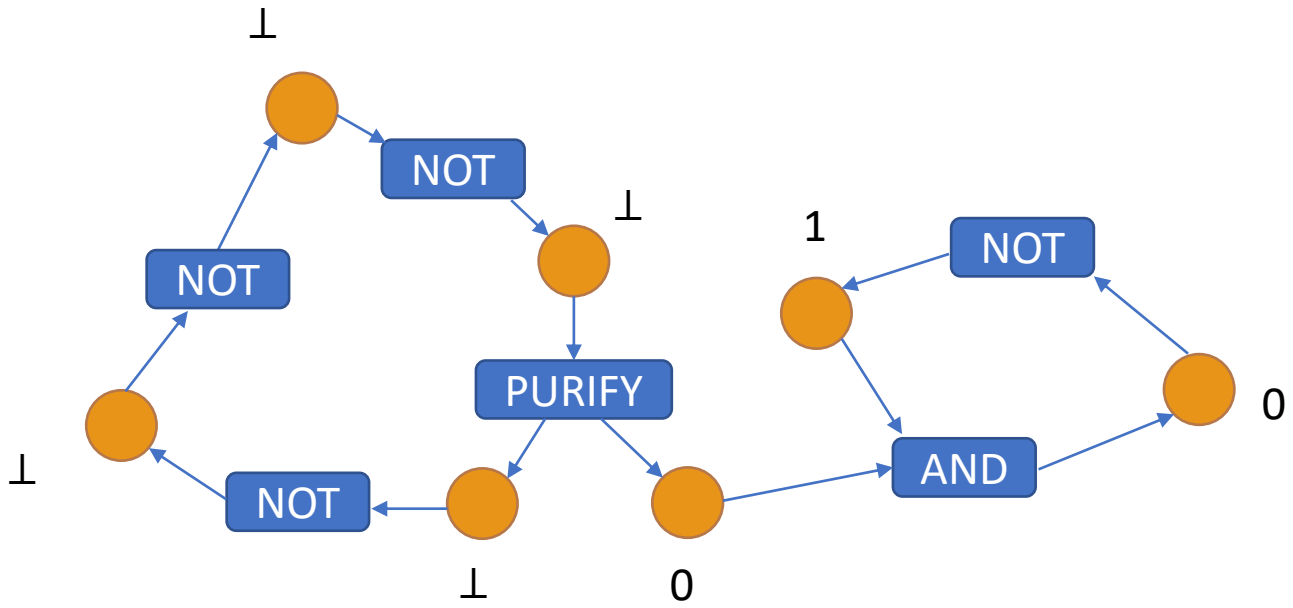

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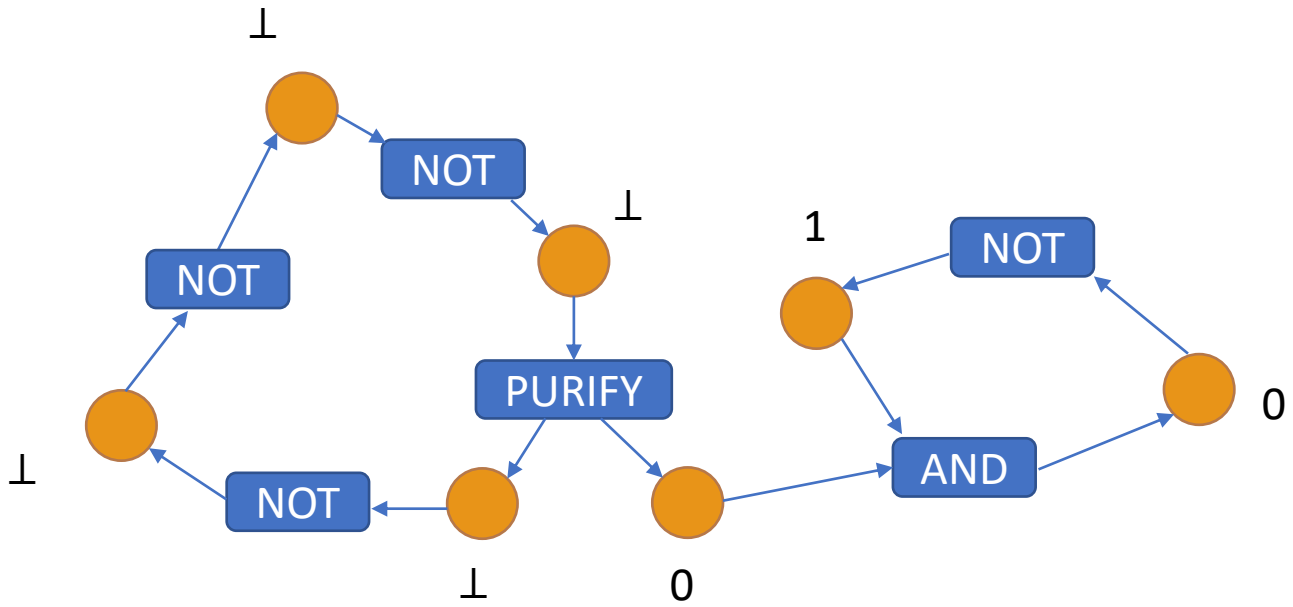

Theorem (with Deligkas, Fearnley, Melissourgos ‘22):
Pure-Circuit is PPAD-complete with gates NOT, AND, PURIFY.

Focus on PPAD

I. How do we prove PPAD-membership?
   1. Sperner's Lemma
   2. The END-OF-LINE Problem
   3. Example: NASH

II. How do we prove PPAD-hardness?
   1. Refresher: Reductions
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   3. **Example: Polymatrix Games**

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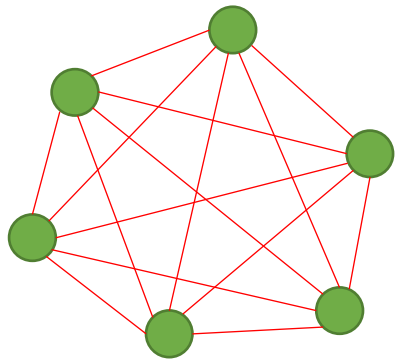

Polymatrix Games: Nash Equilibria

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(Note: additive approximation, smaller ε is better)

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- A simple algorithm for computing a $1/3$-WSNE in a binary-action polymatrix game:

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→ Can show that this always yields a $1/3$-WSNE (by a simple direct computation)

This is a pretty naive algorithm…

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Can we do better?

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Can we do better?
Can we provide evidence that we *cannot* do better?

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For any constant $\varepsilon < 1/3$, computing an ε-WSNE is PPAD-complete.

Proof: For any $\varepsilon < 1/3$, provide a reduction:

$$\text{Pure-Circuit} \leq_p \varepsilon\text{-WSNE-Polymatrix}.$$

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Goal: Construct a polymatrix game

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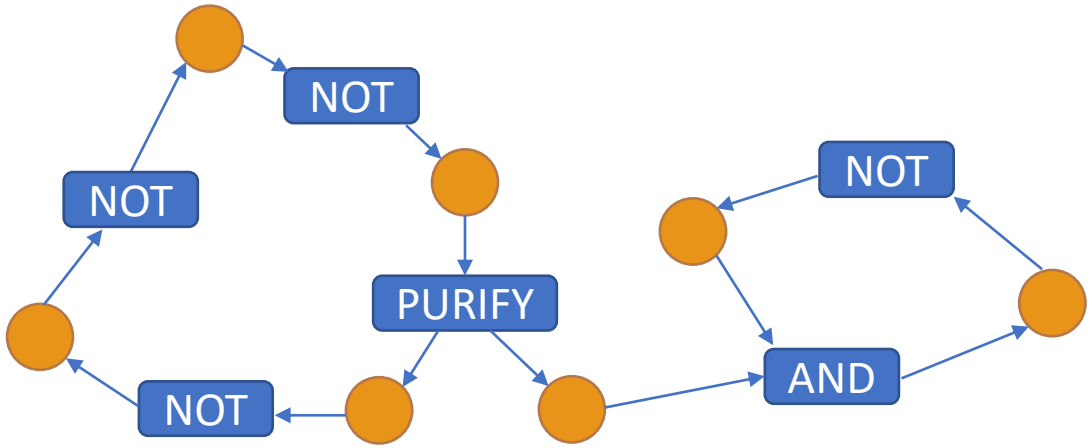

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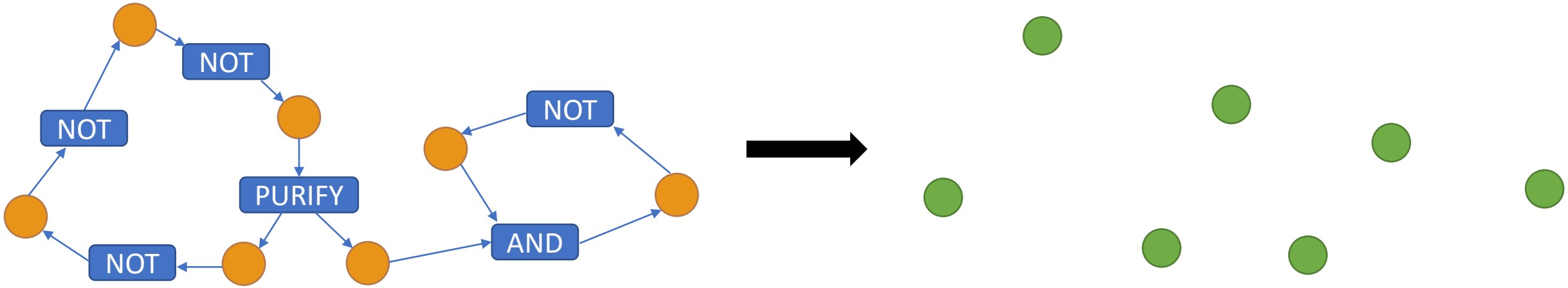

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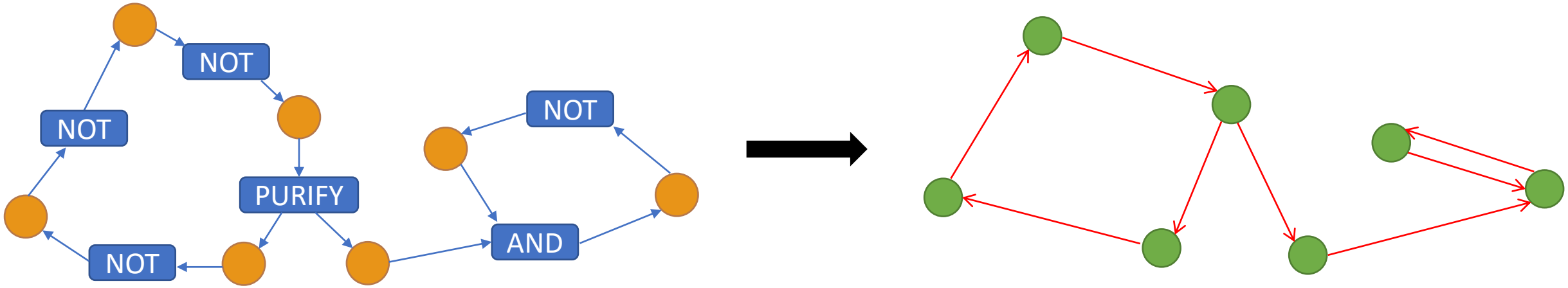
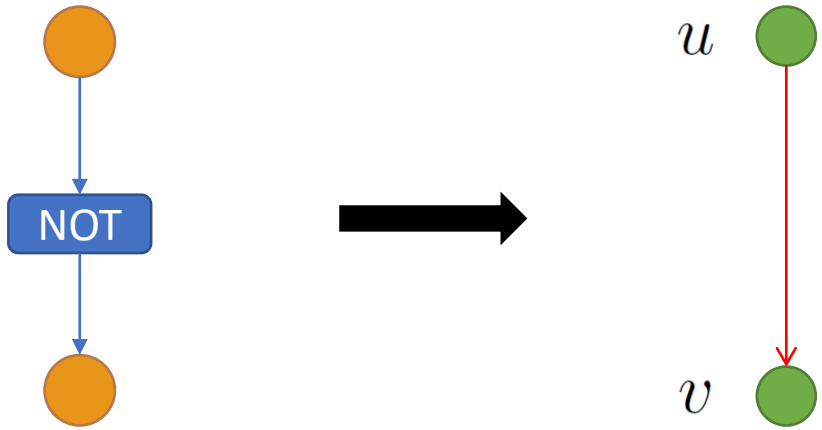


Reduction: NOT-gates



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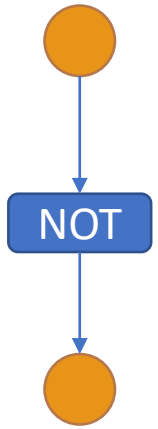

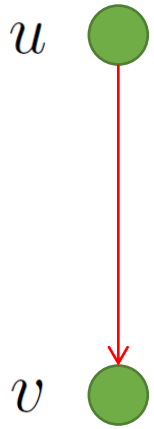

v \ u	zero	one
zero	0 0 0	0 1 0
one	0 1 0	0 0 0

(works for any $\varepsilon < 1$)

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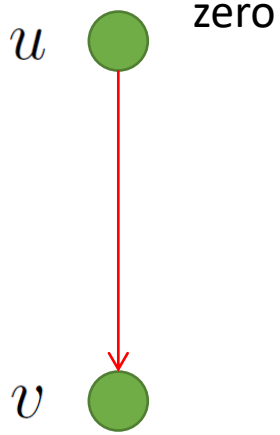

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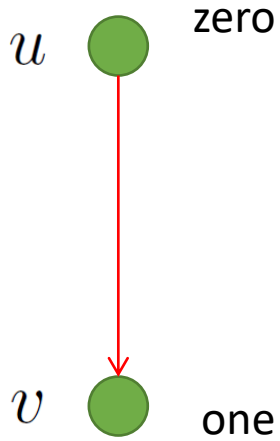

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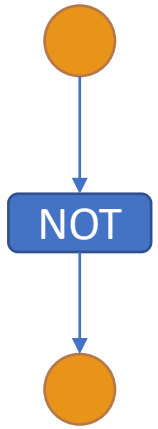

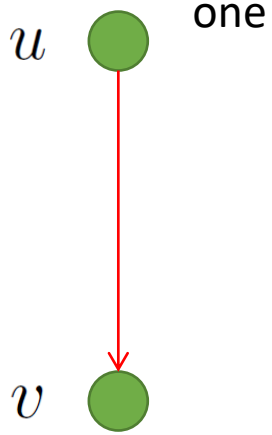

v \ u	zero	one
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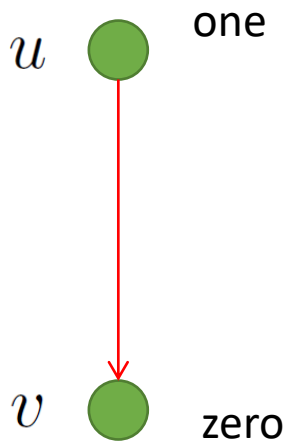
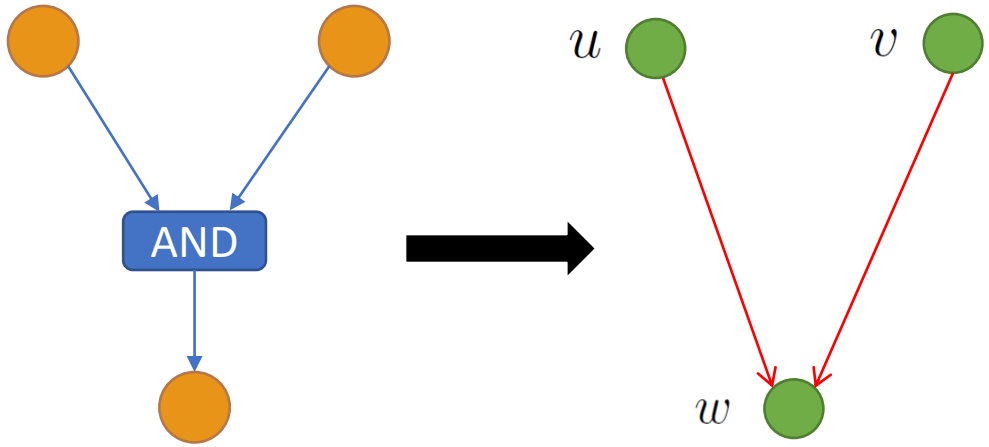


v \ u	zero	one
zero	0 0	0 1
one	0 1	0 0

(works for any $\varepsilon < 1$)

Reduction: AND-gates



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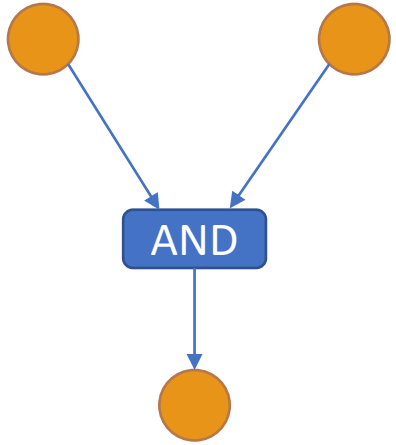
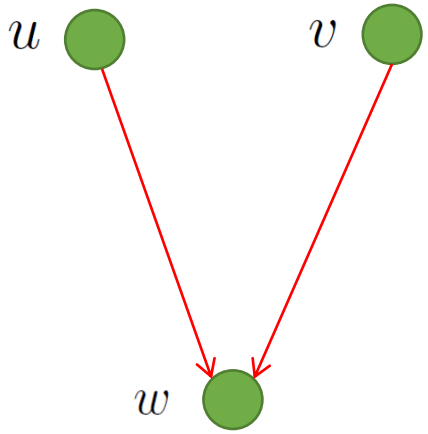


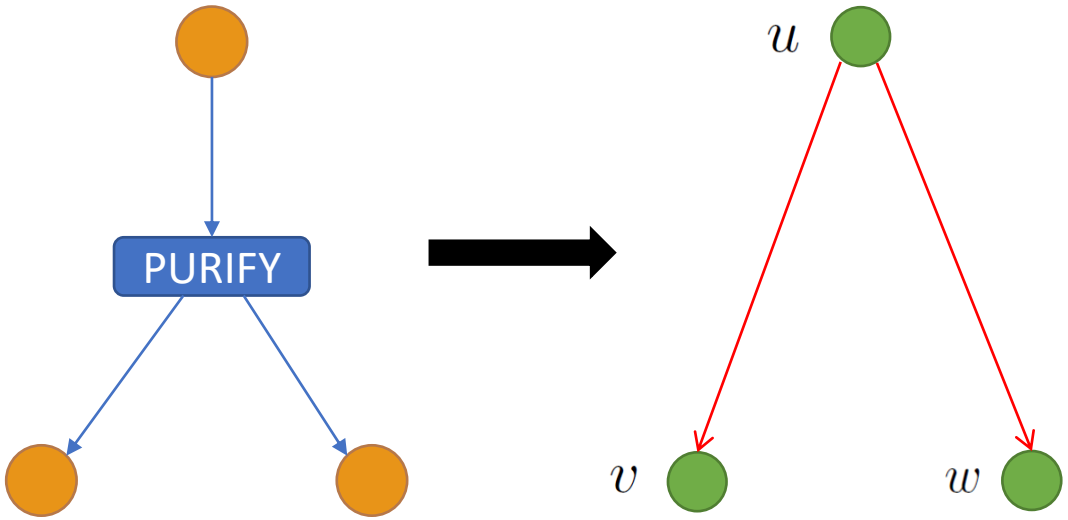


w \ u	zero	one
zero	0, $\frac{1}{2}$	0, 0
one	0, 0	0, $\frac{1}{6}$

w \ v	zero	one
zero	0, $\frac{1}{2}$	0, 0
one	0, 0	0, $\frac{1}{6}$

(works for any $\varepsilon < 1/3$)

Reduction: PURIFY-gates



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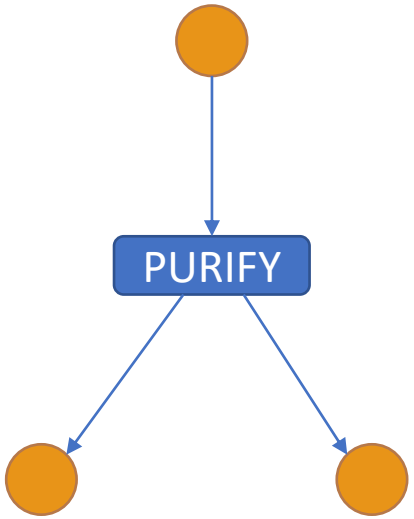

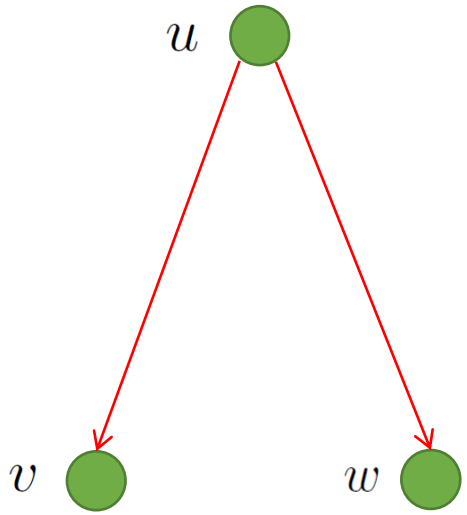

v \ u	zero	one
zero	0, $\frac{1}{3}$	0, 0
one	0, 0	0, 1

w \ u	zero	one
zero	0, 1	0, 0
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From Polymatrix to Bimatrix Games

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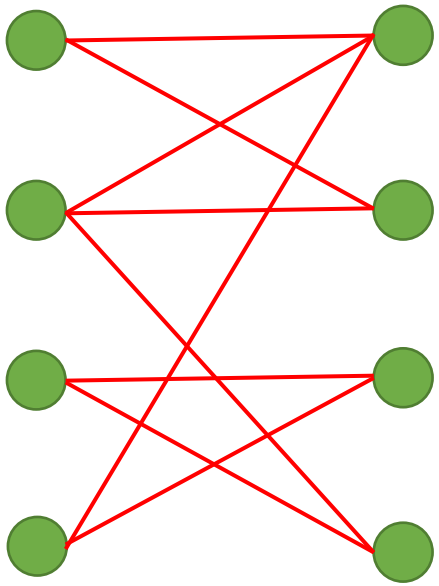

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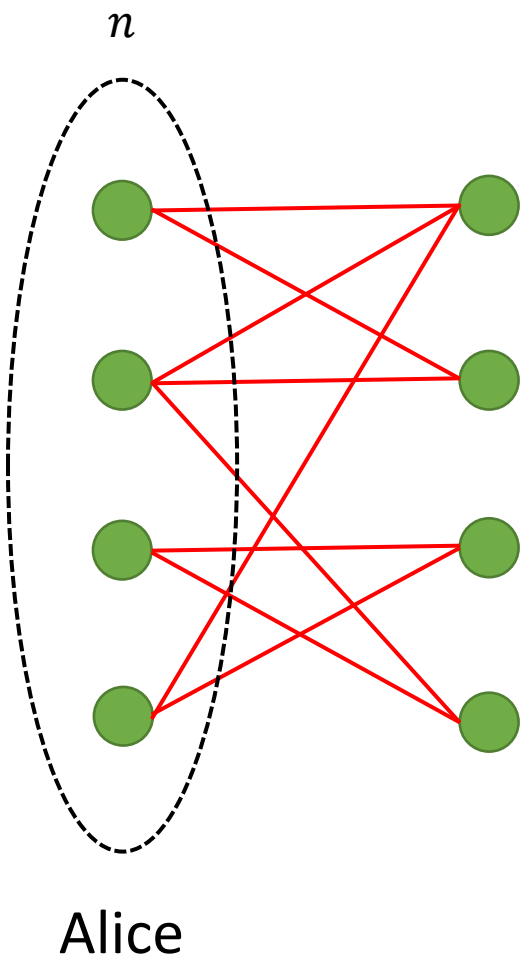

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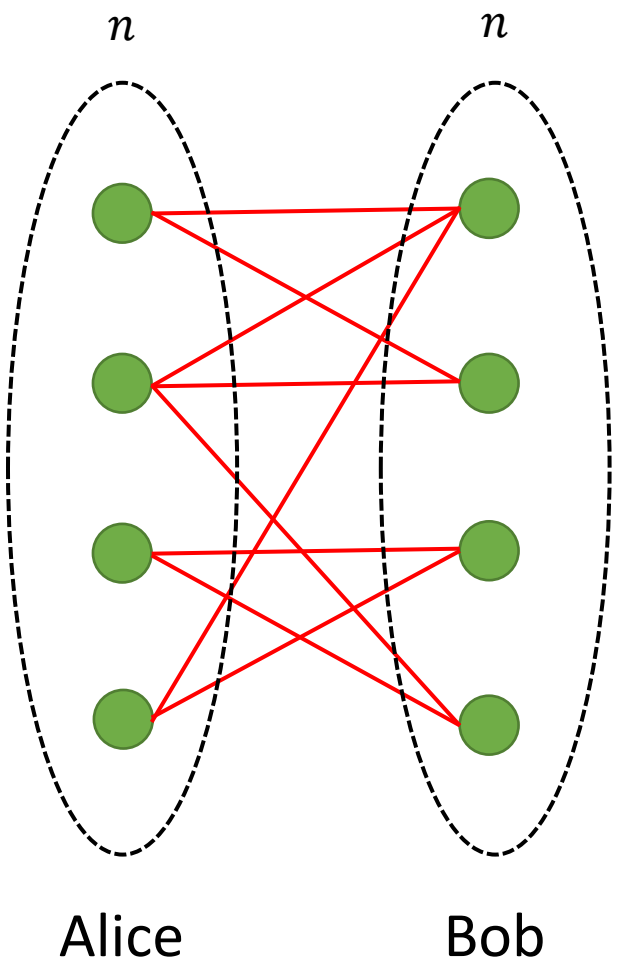
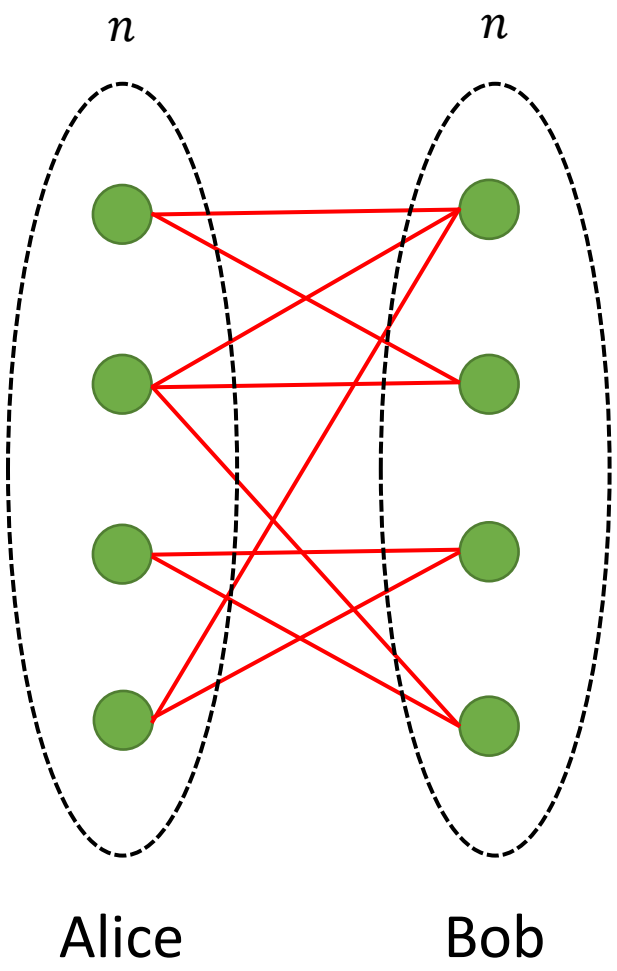
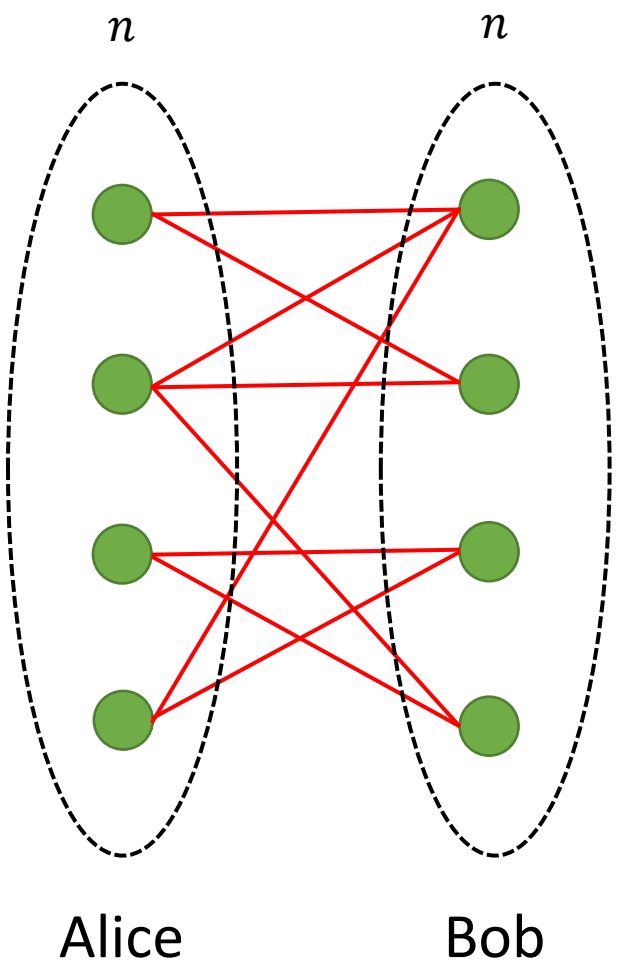


From Polymatrix to Bimatrix Games



Bimatrix game:

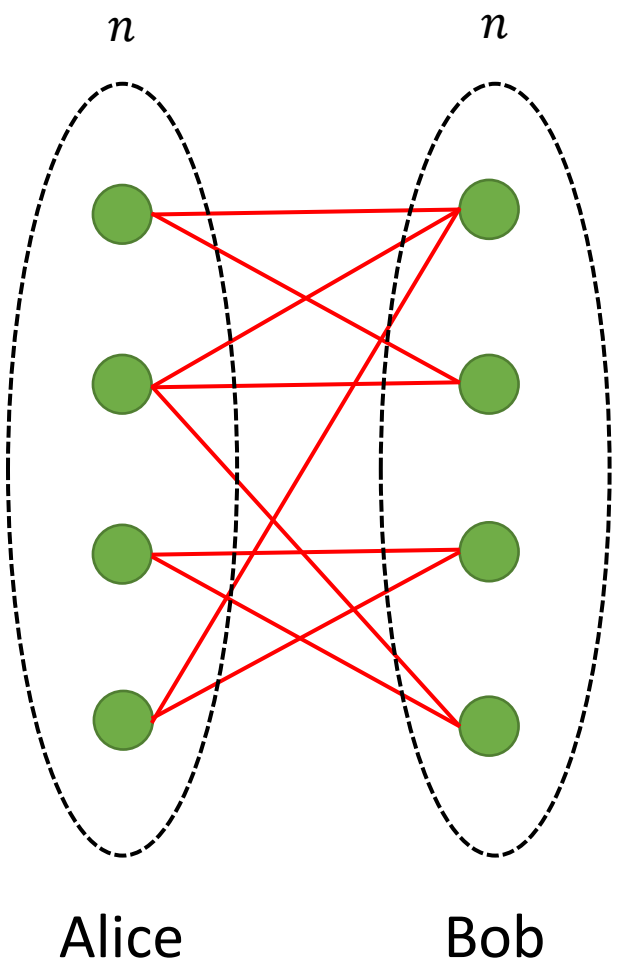
From Polymatrix to Bimatrix Games



Bimatrix game:

- 2 players (Alice and Bob)

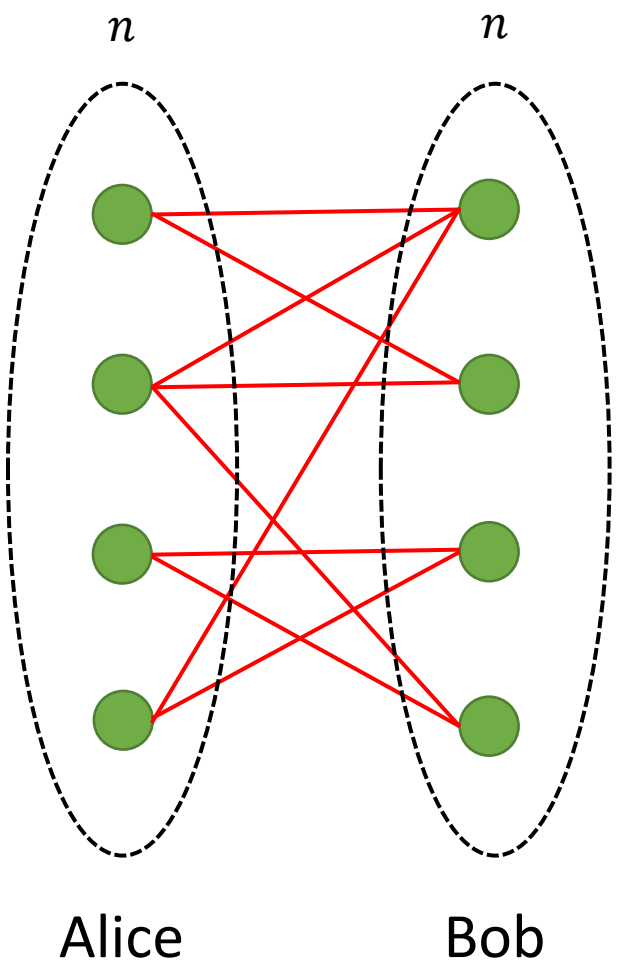
From Polymatrix to Bimatrix Games



Bimatrix game:

- 2 players (Alice and Bob)
- $2n$ actions each

From Polymatrix to Bimatrix Games



Bimatrix game:

- 2 players (Alice and Bob)
- $2n$ actions each

$\varepsilon/p(n)$-equilibrium of bimatrix game
yields ε-equilibrium of polymatrix game

Thank you!