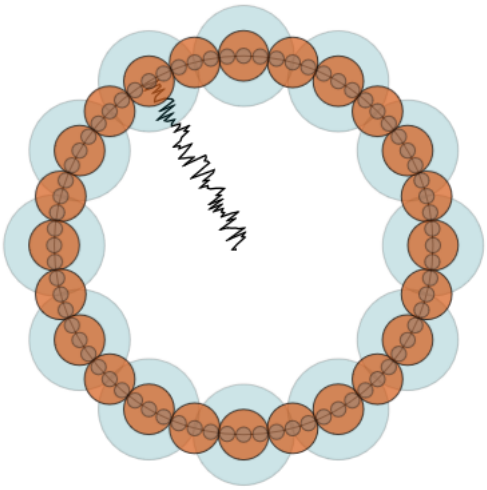
PDEs @ Essex 2025

Regularity Theory of PDEs and Calculus of Variations

Organizer: Murat Akman - murat.akman@essex.ac.uk

16-18 June 2025
University of Essex, Colchester
https://www.essex.ac.uk/events/2025/06/16/pdesatessex2025



This workshop is partially funded by the EPSRC (EP/W001586/1).

Contents

Some Practical Information

The talks will be at **NTC.2.07** (North Teaching Center). https://maps.app.goo.gl/vsF6nBZ3PZEgkEcdA.

Taxi companies in Colchester (from university to town or train station, it would cost about £12 and you can pay cash, with contactless credit cards, Apple/Google Wallet):

- Five sevens taxi (01206 577777)
- Panther Cabs (01206 525525)

You can take bus 87 (University/Great Horkesley), 51 (Wivenhoe/University), or 74 (Clacton) to come to campus if your hotel is in town. (The same buses go back to town). You can pay in cash (£2) or use your Apple/Google Wallet or contactless card. It may take about 30 minutes to arrive on campus. You can also walk, which would take about 45 minutes to 1 hour.

All shops and cafés on campus are cashless — they only accept card payments or mobile payments (Apple Pay, Google Pay, etc.).

You can use eduroam if your home institution supports it. For those without eduroam access, temporary guest Wi-Fi credentials can be obtained through Guest Wifi Network.

The workshop dinner will be at the Wivenhoe House Hotel on the campus on Tuesday (17/06) at 7pm. https://maps.app.goo.gl/DAgAZVnLqGrCKPah8.

Google Maps: Here are a few places of interest on the Google Maps: https://maps.app.goo.gl/4THdZdzbYReo1Rgr5

If you incur any reimbursable expenses (e.g. travel, meals not provided), please keep your receipts, as these will be required for reimbursement.

Workshop Schedule

16 June 2025 (Monday), Room: NTC.2.07.

- 9:00 - 9:30 Registration
- 9:30 - 10:30 **Roger Moser** (University of Bath)
  Title of the talk: Divergence free transition layers in the plane.
- 10:30 - 11:00 Coffee break
- 11:00 - 12:00 **Vahideh Vahidifar** (University of Sussex)
  Title of the talk: Geometry of Nonlinear Diffusion: Gradient Estimates, Entropy Inequalities and Liouville Results on Smooth Metric Measure Spaces.
- 12:00 - 12:25 **Ariel Aguas-Barreno** (University of Essex)
  Title of the talk: The Brunn-Minkowski Inequality for p-Harmonic Measures.
- 12:25 - 14:00 Lunch Break
- 14:00 - 15:00 **Simon Larson** (University of Gothenburg)
  Title of the talk: On a Hardy-Morrey inequality.
- 15:00 - 15:30 Coffee Break
- 15:30 - 16:30 **Max Engelstein** (University of Minnesota-Twin Cities)
  Title of the talk: The Robin Boundary Value Problem in Rough domains.
- 16:30 - 17:30 **Matthew Badger** (University of Connecticut)
  Title of the talk: Nodal Domains of Homogeneous Caloric Polynomials.

17 June 2025 (Tuesday), Room: NTC.2.07.

- 9:30 - 10:30 **Jennifer Duncan** (The Institute of Mathematical Sciences, Spain)
  Title of the talk: Localised multilinear restriction estimates and the helical maximal function.
- 10:30 - 11:00 Coffee Break
- 11:00 - 12:00 **Ana Emilia de Orellana** (University of St Andrews)
  Title of the talk: Fourier restriction and fractal dimensions.
- 12:00 - 12:25 **Cody Michael Hutcheson** (University of Alabama)
  Title of the talk: The Hölder Solvability of the Parabolic Continuous Dirichlet Problem in Rough Sets
- 12:25 - 14:00 Lunch Break
- 14:00 - 15:00 **Martin Dindos** (University of Edinburgh)
  Title of the talk: A relationship between the Dirichlet and Regularity problems for parabolic equations.
- 15:00 - 15:30 Coffee Break
- 15:30 - 16:30 **Chema Martell** (The Institute of Mathematical Sciences, Spain)
  Title of the talk: Solvability of the Dirichlet Problem for Elliptic Operators.
- 16:30 - 17:30 **Steve Hofmann** (University of Missouri)
  Title of the talk: The Dirichlet problem for elliptic equations with a singular drift term.

Workshop Dinner at 19:00 at Wivenhoe House Hotel.

18 June 2025 (Wednesday), Room: NTC.2.07.

- 9:30 - 10:30 **Federica Dragoni** (Cardiff University)
  Title of the talk: Semiconcavity of the square distance in Carnot groups.
- 10:30 - 11:00 Coffee Break
- 11:00 - 12:00 **John Ball** (Heriot-Watt University)
  Title of the talk: Highly irregular microstructures and T_N configurations.
- 12:00 - 12:25 **Arnaud Dumont** (University of Birmingham)
  Title of the talk: Boundary value problems for singular Schrödinger-type equations with block structure.
- 12:25 - 14:00 Lunch Break
- 14:00 - 15:00 **Simon Bortz** (University of Alabama)
  Title of the talk: Green Function Estimates, UR and Elliptic/Parabolic Measure.
- 15:00 - 15:25 **Pablo Hidalgo-Palencia** (The Institute of Mathematical Sciences, Spain)
  Title of the talk: Parabolic Dirichlet problem with Lp data and variable coefficients.
- 15:25 - 15:50 **Marcin Gryszówka** (University of Warsaw)
  Title of the talk: ε-Approximability and Quantitative Fatou Property for a class of non-harmonic functions.
- 15:50 - 16:00 Closing remarks

Titles and Abstracts

The Brunn-Minkowski Inequality for p-Harmonic Measures

16 June
12pm

Ariel Aguas-Barreno
University of Essex

The Brunn-Minkowski inequality is a cornerstone of the Brunn-Minkowski theory. In this talk we will provide a little history surrounding it and we will provide a local p-Harmonic Brunn-Minkowski inequality that is related and a result of the work done by Akman and Mukherjee on the Minkowski Problem for p-Harmonic measures. We will recall the p-Harmonic Minkowski problem and its setting, then provide a decomposition of the p-laplacian into support functions h_u and its derivatives, necessary to our steps of the proof. By proving the p-harmonic Brunn-Minkowski inequality we establish uniqueness for the corresponding p-Harmonic Minkowski problem.

Nodal Domains of Homogeneous Caloric Polynomials

16 June
4:30pm

Matthew Badger
University of Connecticut

With a view towards confirming the existence of singular strata in Mourgoglou and Puliatti's two-phase free boundary regularity theorem for caloric measure, we identify the minimum number of nodal domains of homogeneous caloric polynomials (hcps) in Rn+1 of degree d. We also provide estimates on the maximum number of nodal domains for all n and d. I'll survey the techniques that go into the proofs of the theorems, particularly the construction of hcps that realize the minimum number of nodal domains. This is joint work with Cole Jeznach.

Highly irregular microstructures and T_N configurations

18 June
11am

John Ball
Heriot-Watt University

Remarkable martensitic microstructures are observed in the alloy $Ti_{76}Nb_{22}Al_2$, which undergoes a cubic to orthorhombic transformation with six martensitic variants $\mathbf{U}_i = \mathbf{U}_i^T > 0$ having middle eigenvalue $\lambda_2(\mathbf{U}_i)$ very close to 1. Assuming that $\lambda_2(\mathbf{U}_i) = 1$ there are exactly 12 matrices in the set of energy wells $\bigcup_{i=1}^{6} SO(3)\mathbf{U}_i$ that are rank-one connected to $\mathbf{1}$. This set of 12 matrices has no rank-one connections. We attempt to understand the observed microstructures by studying gradient Young measures, exact gradients and T_N-configurations supported on these 12 matrices. This is joint work with Tomonari Inamura and Francesco Della Porta.

---

18 June
2pm

Green Function Estimates, UR and Elliptic/Parabolic Measure

Simon Bortz

University of Alabama

Together with Hofmann, Martell, and Nyström, we have shown strong connections between parabolic uniform rectifiability and the quantitative absolute continuity of caloric measure. Some of these results were extended by Ferris, Hidalgo-Palencia, Hofmann, and myself to variable-coefficient operators satisfying an L^1 Carleson condition (see P. Hidalgo-Palencia's talk!). A crucial ingredient in this direction is a second-derivative estimate on the Green function in the form of a Carleson measure condition.

Conversely, together with Egert and Saari, we have shown Carleson measure conditions on the (parabolic) Green function in nice domains, which quantify how close it is to being a linear function. This builds on the work of David, Li, and Mayboroda [DLM] in the elliptic setting. The work [DLM] was a key ingredient in establishing certain asymptotic estimates for elliptic measure, shown by Toro, Zhao, and myself, and we extend those results as well. Moreover, we improve an important estimate from [DLM] that enables one to study similar Green function estimates in a uniform manner, obtaining the appropriate bounds whether the coefficients are Hölder continuous or satisfy the rough "weak DKP" condition.

The talk will focus on the work with Egert and Saari, while providing background on the other topics and presenting open questions.

---

17 June
2pm

A relationship between the Dirichlet and Regularity problems for parabolic equations

Martin Dindos

University of Edinburgh

Recall as well-known result of Shen that has established that if $\mathscr{L} = \operatorname{div}(A\nabla\cdot)$ is an elliptic operator on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ then there is a "near duality" between the solvability of the Regularity problem for $\mathscr{L}$ with boundary data $\nabla_T f \in L^p(\partial\Omega)$ and solvability of the Dirichlet problem for the adjoint operator $\mathscr{L}^*$ with boundary data $f \in L^{p'}(\partial\Omega)$ where $1/p + 1/p' = 1$. This duality holds, provided the Regularity problem for $\mathscr{L}$ is solvable for at least one $q \in (1, \infty)$.

In the talk shall present a joint result with Erika Nyström where we have managed to prove analogous result in the parabolic case for an operator of the form $\partial_t - \mathscr{L}$ on parabolic cylinders $\mathscr{O} = \Omega \times \mathbb{R}$. One of key obstacles while tackling this problem was the definition of an end-point atomic Hardy-Sobolev space for the Regularity problem and establishing its interpolation properties with Sobolev spaces. This result is a key ingredient that has subsequently allowed us to resolve the question of solvability of the Regularity problem on Lipschitz cylinders with coefficients satisfying a parabolic analogue of the DKP-condition (in a joint work with Jill Pipher and Linhan Li).

Semiconcavity of the square distance in Carnot groups

18 June
9:30am

Federica Dragoni

Cardiff University

Semiconcavity and semiconvexity are key regularity properties for functions with many applications in a broad range of mathematical subjects. The notions of semiconcavity and semiconvexity have been adapted to different geometrical contexts, in particular in sub-Riemannian structures such as Carnot groups, where they turn out to be extremely useful for the study of solutions of degenerate PDEs.

In this talk I will show that, for a suitable class of Carnot groups, the Carnot-Carathéodory distance is semiconcave, in the sense of the group, in the whole space.

I will also give some applications to solutions of non-coercive Hamilton-Jacobi equations.

Joint work with Qing Liu and Ye Zhang from OIST (Okinawa, Japan).

Boundary value problems for singular Schrödinger-type equations with block structure

18 June
12pm

Arnaud Dumont

University of Birmingham

The study of boundary value problems in the upper half-space for divergence-form elliptic equations with block structure and complex coefficients independent of the transversal direction to the boundary has recently been settled by P. Auscher and M. Egert, who streamlined the methods developed over the last two decades to solve such problems for many boundary data spaces, including those of L^p-type, in their recent monograph.

In this talk we shall describe how this machinery can be adapted to the study of Dirichlet and regularity boundary value problems for a class of singular Schrödinger-type elliptic equations in the upper half-space, by relying on some essential L^2 estimates recently obtained by A. Morris and A. Turner for a class of Schrödinger operators with reverse-Hölder class potentials.

This talk is based on joint work with Andrew Morris (University of Birmingham, UK).

Localised multilinear restriction estimates and the helical maximal function

17 June
9:30am

Jennifer Duncan

The Institute of Mathematical Sciences, Spain

A fundamental result in modern fourier analysis is the celebrated Bennett–Carbery–Tao multilinear restriction estimate: given an ensemble $(S_1,\dots,S_n)$ of compact transversal hypersurfaces in $\mathbb{R}^n$, let E_j denote the fourier extension operator associated with S_j for each $1 \le j \le n$. Then, for all $\varepsilon, R > 0$,

$$\left\| \prod_{j=1}^{n} E_j f_j \right\|_{L^{\frac{2}{n-1}}(B_R)} \lesssim_\varepsilon R^\varepsilon \prod_{j=1}^{n} \|f_j\|_{L^2}.$$

This inequality has found use in a number of important contexts, examples including the linear restriction problem, decoupling theory, and the theory of maximal functions, to name a few. In recent years, certain localised variants of multilinear restriction estimates have been found to greatly extend the utility of this framework. In this talk, I will describe the localised multilinear restriction estimate and discuss the main aspects of its proof, in particular how the underlying geometry is captured by a certain generalised notion of a Brascamp–Lieb inequality. I will then illustrate how these localised multilinear restriction estimates may be applied in conjunction with other tools from harmonic analysis in order to establish the complete range (up to endpoints) of off-diagonal estimates for the (local) helical maximal function in $\mathbb{R}^3$, which we define as, given a curve $\gamma : [-1,1] \to \mathbb{R}^3$,

$$M_\gamma f(x) := \sup_{t\in[1,2]} \left| \int_{\mathbb{R}} f(x - t\gamma(s))\chi(s)ds \right|,$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a bump function supported on $[-1,1]$, and we assume that γ has non-vanishing torsion. This is joint work with David Beltran and Jonathan Hickman.

---

17 June
11am

Fourier restriction and fractal dimensions

Ana Emilia de Orellana
University of St Andrews

Fourier restriction is a field of harmonic analysis that establishes the connection between Fourier analysis and the geometric properties of measures, and it has applications to several topics in PDEs. In this talk we will introduce the Fourier restriction problem and see an extension of the Stein-Tomas theorem to the L^q dimensions: a family of dimensions that capture multifractal behaviour of measures.

---

16 June
3:30pm

The Robin Boundary Value Problem in Rough domains

Max Engelstein
University of Minnesota-Twin Cities

Motivated by questions of solvability and also by a PDE model of human lungs, we investigate solutions to elliptic PDE with Robin boundary conditions in rough domains. We construct, for all purely second order elliptic operators in a large class of domains, the Robin harmonic measure. Surprisingly, and in contrast with the Dirichlet problem, we show that this measure is absolutely continuous with respect to surface measure, even when the operator and domain are extremely rough.

In work in progress, we explain this phenomenon further, showing that in some sense Robin solutions are an averaging of Dirichlet solutions. A consequence of this is that we can rigorously demonstrate certain observed and numerically simulated properties of mammalian lungs. All of this is joint work with G. David, S. Decio, S. Mayboroda and M. Michetti.

ε-Approximability and Quantitative Fatou Property for a class of non-harmonic functions

18 June
3:25pm

Marcin Gryszówka

University of Warsaw

ε-Approximability and Quantitative Fatou Property were studied for functions which are solutions of divergence type PDEs. In our work we consider a class of non-harmonic functions, which do not have to be solutions to any PDEs, and prove these two properties. ε-Approximability means that for a certain function there exists its approximant which is epsilon close to it in supremum norm and such that the norm of its gradient gives rise to a Carleson measure. As a consequence, once we know that a function is ε-approximable, we can prove Quantitative Fatou Property. It means that the function counting oscillations of a function is locally integrable and its integral is bounded by the measure of a boundary of a domain. Notice that the classical Fatou theorem may be stated in the following way: the function counting oscillations is finite a.e. Hence, Quantitative Fatou Property is stronger. The talk is based on a joint work with T. Adamowicz and M.J. González.

Parabolic Dirichlet problem with Lp data and variable coefficients

18 June
3pm

Pablo Hidalgo-Palencia

The Institute of Mathematical Sciences, Spain

It is a very recent result of Bortz, Hofmann, Martell and Nyström (in combination with earlier works of Hofmann, Lewis and Murray) that the Dirichlet problem for the heat equation is well-posed in Lp in domains whose boundary is a (parabolic) Lipschitz graph, with the additional property that the graph has a half-time derivative in BMO. They show that this condition, although seemingly artificial at first glance, is actually optimal.

In this talk, we show an extension of this result beyond the heat equation. Concretely, we are able to prove that, for a family of coefficients satisfying a Carleson measure condition "of L1 type", it is necessary that half-time derivatives are in BMO in order to obtain well-posedness of the Dirichlet problem.

This is joint work with Simon Bortz, Sandra Ferris and Steve Hofmann.

---

17 June
4:30pm

The Dirichlet problem for elliptic equations with a singular drift term

Steve Hofmann

University of Missouri

We establish solvability of the L^p Dirichlet problem, for some $p < \infty$, for elliptic equations of the form

$$Lu := -\mathrm{div}\,(A\nabla u) + \mathbf{b}\cdot\nabla u =: L_0 u + \mathbf{b}\cdot\nabla u = 0\,,$$

in a 1-sided chord arc domain $\Omega \subset \mathbb{R}^{n+1}$ (i.e., a uniform domain with an Ahlfors-David regular boundary), provided that L^p solvability holds (typically for a different p) for the homogeneous second order equation $L_0 u = 0$, and (roughly speaking) that the drift term satisfies $\mathrm{dist}(X,\partial\Omega)|\mathbf{b}(X)| \lesssim 1$, as well as the Carleson measure condition

$$\iint_{\Omega\cap B(x,r)} |\mathbf{b}(Y)|^2 \mathrm{dist}(Y,\partial\Omega)\,dY \lesssim r^n, \quad x\in\partial\Omega\,, 0<r<\mathrm{diam}(\partial\Omega).$$

In previous work with J. L. Lewis, we had treated the analogous problem for parabolic equations in the half-space (and hence, via a pullback mechanism, in certain parabolic Lipschitz graph domains), and had presented a claimed, simpler proof in the elliptic case, based on an erroneous proof of doubling of elliptic measure for operators with drift terms. In fact, it appears that doubling remains an open problem (except in the small constant case), so the putative elliptic argument in that work was incorrect.

The proof presented in the present work provides a purely elliptic argument, still simpler than that in the parabolic case, which does not require doubling (again, except in the small constant case). Moreover, the present argument allows the treatment of much more general domains, whose boundaries need not be given locally as graphs.

---

17 June
12pm

The Hölder Solvability of the Parabolic Continuous Dirichlet Problem in Rough Sets

Cody Michael Hutcheson

University of Alabama

We prove the Hölder solvability of the parabolic continuous Dirichlet problem in rough sets, which is an extension of the main result of a paper by Cao, Hidalgo-Palencia, Martell, Prisuelos-Arribas, and Zhao. To do so, we generalize a work by Genschaw and Hofmann; namely, we weaken their ADR condition to a Hausdorff content condition, which is similarly operator independent. In addition, we show that these results can be extended from C^∞ coefficient operators to L^∞ ones.

On a Hardy-Morrey inequality

16 June
2pm

Simon Larson
University of Gothenburg

In this talk I will discuss a Hardy-type inequality on domains in Euclidean space. In particular, we will be interested in how sharp constants depend on the underlying geometry and whether or not extremals exist. The talk is based on joint work with Ryan Hynd (UPenn) and Erik Lindgren (KTH).

Solvability of the Dirichlet Problem for Elliptic Operators

17 June
3:30pm

Chema Martell
The Institute of Mathematical Sciences, Spain

In this talk, we will explore the geometric conditions that ensure the solvability of Dirichlet problems for the Laplacian, and more generally, for second-order divergence-form elliptic operators. These conditions are closely related to the nature of the boundary data, and we will focus in particular on the case where the boundary data satisfies Hölder continuity conditions. Our goal is to identify the minimal geometric assumptions on the domain that guarantee solvability of the Dirichlet problem in this setting.

Divergence free transition layers in the plane

16 June
9:30am

Roger Moser
University of Bath

For vector fields in a two-dimensional domain, consider a Modica-Mortola (or Allen-Cahn) type functional. We do not make any specific assumptions on the wells of the potential function (so there may be multiple single-point wells or one or several more complex wells), but we do assume that the divergence of the vector fields is quite strongly penalised or even vanishes identically.

As in other Modica-Mortola type problems, when we let the relevant parameter tend to 0, we expect a limit that takes values in the wells of the potential function, but there can be transitions between different values. We want to find out how much energy is required for such a transition.

Our strategy is to find suitable functions, called calibrations, which measure the energy of a transition layer. This is not a new idea, but we take it in a somewhat different direction. When asking, what is the optimal calibration for a given potential function, we are led to a variational problem involving the L-infinty norm of the gradient. Such problems are difficult to study, but using recent ideas of Katzourakis and myself, we can encapsulate the essential information in a geometric variational problem. Solving the latter can still be difficult, but we finally obtain an answer at least in some cases.

This is joint work with Radu Ignat (Institut de mathématiques de Toulouse).

---

16 June
11am

Geometry of Nonlinear Diffusion: Gradient Estimates, Entropy Inequalities and Liouville Results on Smooth Metric Measure Spaces

Vahideh Vahidifar

University of Sussex

In this talk I will present gradient estimates of Hamilton-Souplet-Zhang and Li-Yau types for a class of nonlinear diffusion equations on smooth metric measure spaces. The Laplace-Beltrami operator gives its place to the Witten Laplacian and the Riemannian metric tensor and potential evolve with time (a geometric flow). The estimates are established under different curvature conditions and lower bounds on the Bakry-Emery Ricci tensor and are then used to prove a number of important results such as Harnack inequalities, spectral bounds, sharp Logarithmic Sobolev inequalities (LSI) and general Liouville and global constancy results. If time allows, I will present applications of the above to entropy dissipation inequalities and geometric deep learning.
