PDEs @ Essex 2024

Problems in potential theory of nonlinear PDEs

Organizer: Murat Akman - murat.akman@essex.ac.uk

20-22 February 2024
University of Essex, Colchester
https://www.essex.ac.uk/events/2024/02/20/PDEsatEssex2024

Some Practical Information

All the talks will be at the Essex Business School (EBS.2.41). https://maps.app.goo.gl/yFEQwpSj5wz5W1jSA

The school recently became a node of the LMS Harmonic Analysis and PDE network: https://sites.google.com/view/lms-hanpde-network/home, and the talks on Tuesday (February 20) are to celebrate this.

Taxi companies in Colchester (from university to town or train station, it would cost about £12 and you can pay cash, with contactless credit cards, Apple/Google Wallet):

- Five sevens taxi (01206 577777)
- Panther Cabs (01206 525525)

You can take bus 87 (University/Great Horkesley), S1 (Wivenhoe/University), or 74 (Clacton) to come to campus if your hotel is in town. (The same buses go back to town). You can pay in cash (£2) or use your Apple/Google Wallet or contactless card. It may take about 30 minutes to arrive on campus. You can also walk, which would take about 45 minutes to 1 hour.

If you stay at the Wivenhoe House Hotel, it is on the campus and you can walk to the Essex Business School (EBS.2.41). It would not take more than 20 minutes.

The workshop dinner will be at the Wivenhoe House Hotel on the campus on Wednesday (21/02) at 7pm. https://maps.app.goo.gl/DAgAZVnLqGrCKPah8.

Google Maps: Here are a few places of interest on the Google Maps: https://maps.app.goo.gl/UNBZXjbLn1PbvNMu9

Workshop Schedule

20 February 2024 (Tuesday) – LMS HAnPDE network meeting

- 14:00 - 14:50 Linhan Li - University of Edinburgh
  Title of the talk: A Green function characterization of uniform rectifiability of any codimension

- 15:00 - 15:50 Shirsho Mukherjee - University of Essex
  Title of the talk: Minkowski problem for p-harmonic measures

- 15:50-16:20 Coffee Break

- 16:20 - 17:10 Peter Van Hintum - University of Oxford
  Title of the talk: Sharp stability for the Brunn-Minkowski inequality for arbitrary sets

21 February 2024 (Wednesday)

- 9:00 - 9:30 Coffee
- 9:30 - 10:20 Kaj Nyström - Uppsala University (Sweden)
  Title of the talk: Solvability of the L^p Dirichlet problem for the heat equation and parabolic uniform rectifiability
- 10:30 - 11:20 Andrea Pinamonti - University of Trento (Italy)
  Title of the talk: Some regularity results for balance laws and applications to the Heisenberg group.
- 11:20 - 11:50 Coffee break
- John Lewis - University of Kentucky (USA)
  Title of the talk: What I learned from Professor Walter Hayman
- 12:40 - 14:10 Lunch Break
- 14:10 - 15:00 Juan Manfredi - University of Pittsburgh (USA)
  Title of the Talk: On the ∞-Laplacian on Carnot Groups
- 15:00 - 15:30 Coffee Break
- 15:30 - 16:20 George Papamikos - University of Essex
  Title of the talk: Symmetries, reductions and solutions of some elliptic PDEs and related equations
- 16:30 - 17:20 Alexander Pushnitski - King's College London
  Title of the talk: Optimal regularity for the cubic Szegő equation
- 19:00 - 22:00 Workshop Dinner at the Wivenhoe House Hotel

21 February 2024 (Wednesday)

- 9:00 - 9:30 Coffee
- 9:30 - 10:20 Ali Taheri - University of Sussex
  Title of the talk: Curvature conditions, Gradient estimates on the Witten Laplacian and Liouville theorems on smooth metric measure spaces
- 10:30 - 11:20 Erik Sätterqvist - University of Edinburgh
  Title of the talk: A Relation Between the Dirichlet and Regularity Problem for Parabolic PDEs
- 11:20 - 11:50 Coffee Break
- 11:50 - 12:40 Jan Kristensen - University of Oxford
  Title of the talk: TBA
- 12:40 - 14:10 Lunch Break
- 14:10 - 15:00 Bruno Poggi - Universitat Autònoma de Barcelona (Spain)
  Title of the talk: Two problems for the magnetic Schrödinger operator and their solutions via the landscape function.
- 15:00 - 15:30 Coffee Break
- 15:30 - 16:20 Pablo Hidalgo Palencia - Instituto de Ciencias Matemáticas (Spain)
  Title of the talk: Perturbation of elliptic operators in domains without connectivity

Contents

Titles and Abstracts

Sharp stability for the Brunn-Minkowski inequality for arbitrary sets

20 Feb
4:20pm

Peter Van Hintum
University of Oxford

The Brunn-Minkowski inequality states that for (open) sets A and B in $\mathbb{R}^d$, we have

$$|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}.$$

Equality holds if and only if A and B are convex and homothetic sets in $\mathbb{R}^d$. In this talk, I'll present a sharp stability result for the Brunn-Minkowski inequality, concluding a long line of research on this problem. We show that if A and B are close to equality in the Brunn-Minkowski inequality, then they are close to being homothetic and convex, establishing the exact dependency between the three notions of closeness. This talk is based on joint work with Alessio Figalli and Marius Tiba.

The Burkholder functional on classes of planar quasiregular maps

22 Feb
11:50am

Jan Kristensen
University of Oxford

The area formula of Gronwall and Bieberbach can be viewed as a precise way to express that the Jacobian of a planar

Sobolev map is a null Lagrangian. In this talk I discuss a quasiconvexity inequality for the Burkholder functional in the context of

planar quasiconformal maps. This inequality can be viewed as an extension of the area formula to an Lp context. If combined

with Stoilow factorization and blow-up arguments it also allows a proof of semicontinuity, and hence to prove existence of

minimizers for the Burkholder energy on classes of planar quasiregular maps. The talk is based on joint work with Kari Astala (Helsinki),

Daniel Faraco (Madrid), Andre Guerra (ETH), and Aleksis Koski (Aalto).

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21 Feb
11:50am

What I learned from Professor Walter Hayman

John Lewis
University of Kentucky (USA)

During the Covid pandemic I decided to have another look at a problem which I made some progress on during my early mathematical career with the help of Walter Hayman. To state this problem, let $n \geq 2$ be a positive integer, $x = (x_1, x_2, \ldots, x_n)$ a point in Euclidean n space, $\mathbb{R}^n$, and let $|x|$ denote the norm of x. Put $B(x, r) = \{x : |x| < r\}$ when $r > 0$. For fixed $n \geq 2$, let μ be a positive Borel measure on $\mathbb{S}^{n-1} = \{x : |x| = 1\}$ with $\mu(\mathbb{S}^{n-1}) = 1$. Fix $0 \leq d \leq 1 \leq M$, and let $\mathcal{F}_d^M$ denote the family of potentials p with $d \leq p$ in $\bar{B}(0, 1)$ and $P \leq M$ in $\mathbb{R}^n$, satisfying

$$(a) \quad p(x) = \int_{\mathbb{S}^{n-1}} |x - y|^{2-n} d\mu(y), x \in \mathbb{R}^n, \text{ when } n > 2,$$
$$(b) \quad p(x) = 2\int_{\mathbb{S}^{n-1}} \log \frac{1}{|x - y|} \, d\mu(y), x \in \mathbb{R}^2.$$

Let $\mathcal{H}^{n-1}$ denote surface area on $\mathbb{S}^{n-1}$ and let Φ be an increasing convex function on $\mathbb{R}$.

Conjecture: If $\mathcal{F}_d^M \neq \emptyset$, then for $0 < r < \infty$,

$$\int_{\mathbb{S}^{n-1}} \Phi(p(ry)) d\mathcal{H}^{n-1} y \leq \int_{\mathbb{S}^{n-1}} \Phi(P(ry)) d\mathcal{H}^{n-1} y \tag{1}$$

where $P = P(\cdot, d, M) \in \mathcal{F}_d^M$ is unique up to a rotation of $\mathbb{S}^{n-1}$ (independently of Φ) and defined as follows: There exist (also unique) α, β with $-1 \leq \beta \leq \alpha \leq 1$, and

$$\begin{aligned} &(a) \quad P(\cdot, d, M) \equiv M \text{ on } E_1 = \{x \in \mathcal{S}^{n-1} : \alpha \leq x_1 \leq 1, \} \\ &(b) \quad P(\cdot, d, M) \equiv d \text{ on } E_2 = \{x \in \mathbb{S}^{n-1} : -1 \leq x_1 \leq \beta\} \\ &(c) \quad P(0) = 1 \text{ and } \ P(\cdot, d, M) \text{ is harmonic in } \mathbb{R}^n \setminus (E_1 \cup E_2). \end{aligned} \tag{2}$$

During my talk I will discuss some special cases of this Conjecture, together with reminiscences of Professor Walter Hayman. Finally time permitting I will outline the proof of (2) (an obvious necessary condition for the validity of (1).

A Green function characterization of uniform rectifiability of any codimension

20 Feb
2:00pm

Linhan Li
University of Edinburgh

In this talk, I will present a unified characterization of rectifiability of a set of any codimension in terms of the Green function for a divergence-form elliptic operator. This result is built upon earlier works on the Green function and a regularized distance to the set, which I will also discuss. This talk is mainly based on joint work with Joseph Feneuil.

On the ∞-Laplacian on Carnot Groups

21 Feb
2:10pm

Juan Manfredi
University of Pittsburgh (USA)

We study Lipschitz regularity of viscosity solutions to non-homogeneous equations of the form

$$\Delta_\infty^{\mathbb{G}} u = \sum_{i,j=1}^{m} X_i X_j u \, X_i u \, X_j u = f,$$

on a Carnot group $\mathbb{G}$ with horizontal layer $\mathfrak{X} = \{X_1, X_2, \dots, X_m\}$, where f is a, possibly sign changing, real valued continuous function.

We also consider uniqueness of viscosity solutions in the homogeneous case $f = 0$.

This presentation is based on joint work with Fausto Ferrari (Bologna), Nicolò Forcillo (Bologna), and Shirsho Mukherjee (Essex).

Minkowski problem for p-harmonic measures

20 Feb
3:00pm

Shirsho Mukherjee
University of Essex

An account of Minkowski problems shall be discussed including a brief history and contemporary developments followed by our recent results in this direction corresponding to p-harmonic measures. This is a joint work with Murat Akman.

Solvability of the L^p Dirichlet problem for the heat equation and parabolic uniform rectifiability

21 Feb
9:30am

Kaj Nyström
University of Essex

In recent works we solve two problems concerning the L^p Dirichlet problem for the heat equation, caloric measure, and parabolic uniform rectifiability. In [BHMN1] we prove that solvability of the L^p Dirichlet problem is equivalent to parabolic uniform rectifiability in the case of a parabolic Lipschitz graph. In [BHMN2] we prove, in the context of parabolic Ahlfors-David regular boundaries, that solvability of the L^p Dirichlet problem implies parabolic uniform rectifiability. The purpose of the talk is to briefly discuss these results.

[BHMN1]: Simon Bortz, Steve Hofmann, Jose Maria Martell, and Kaj Nyström. Solvability of the L^p Dirichlet problem for the heat equation is equivalent to parabolic uniform rectifiability in the case of a parabolic Lipschitz graph, submitted, 2023.

[BHMN2]: Simon Bortz, Steve Hofmann, Jose Maria Martell, and Kaj Nyström. Solvability of the L^p Dirichlet problem for the heat equation implies parabolic uniform rectifiability, preprint, 2024.

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22 Feb
3:30pm

Perturbation of elliptic operators in domains without connectivity

Pablo Hidalgo Palencia

Instituto de Ciencias Matemáticas (Spain)

Many authors have shown in the last decades that good properties of elliptic PDE in a domain (like solvability of the Dirichlet problem in some Lp) are intimately related to its geometry (e.g. its boundary being uniformly rectifiable). In this talk, we will show that some of this connections even hold under very mild background hypotheses: concretely, with no assumption on connectivity in our domain. These connections allow us to show that some well-known properties, like Carleson measure estimates for bounded solutions, are stable under perturbations of the operator. This is based in a joint work with M. Cao and J.M. Martell.

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21 Feb
3:30pm

Symmetries, reductions and solutions of some elliptic PDEs and related equations

George Papamikos

University of Essex

TBA

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21 Feb
10:30am

Some regularity results for balance laws and applications to the Heisenberg group

Andrea Pinamonti

University of Trento (Italy)

In this talk we prove Hölder regularity of any continuous solution u to a 1d scalar balance law, when the source term is bounded and the flux is nonlinear of order $p \in N$ with $p \geq 2$. Moreover, we prove that at almost every point (t, x) it holds

$$u(t, x + h) - u(t, x) = o(|h|^{\frac{1}{p}}) \text{ as } h \to 0.$$

We apply the results to provide a new proof of the Rademacher theorem for intrinsic Lipschitz functions in the first Heisenberg group. The talk is based on a joint paper with L. Caravenna and E. Marconi.

Two problems for the magnetic Schrödinger operator and their solutions via the landscape function.

22 Feb
2:10pm

Bruno Poggi

Universitat Autónoma de Barcelona (Spain)

In two papers in the 90's, Zhongwei Shen studied non-asymptotic bounds for the eigenvalue counting function of the magnetic Schrödinger operator, as well as the localization of eigenfunctions. But in dimensions 3 or above, his methods required a strong quantitative assumption on the gradient of the magnetic field; in particular, this excludes many singular or irregular magnetic fields, and the questions of treating these later cases had remained open, giving rise to a problem and a conjecture in this area at the intersection of harmonic analysis, mathematical physics, and partial differential equations. In this talk, we present our solutions to these questions, and other new results on the exponential decay of solutions (eigenfunctions, integral kernels, resolvents) to Schrödinger operators. We will introduce the Filoche-Mayboroda landscape function for the (non-magnetic) Schrödinger operator, present its pointwise equivalence to the classical Fefferman-Phong-Shen maximal function (also known as the critical radius function in harmonic analysis literature), and then show how one may use directionality assumptions on the magnetic field to construct a new landscape function in the magnetic case. We resolve the problem and the conjecture of Z. Shen (and recover other results in the irregular setting) by putting all these observations together.

Optimal regularity for the cubic Szegö equation

21 Feb
4:30pm

Alexander Pushnitski

King's College London

The talk is based on recent joint work with Patrick Gérard (Paris-Orsay). The cubic Szegö equation was introduced by Gérard and Grellier in 2010 as a model for totally non-dispersive evolution PDE. It is a Hamiltonian evolution equation on functions from the Hardy class H^2 of analytic functions in the unit disk. Subsequently, Gérard and Grellier have discovered a Lax pair and the action-angle variables for this equation. Using the Lax pair structure, it became possible to prove several optimal results that are usually beyond reach for other equations. The talk will be devoted to one of such results: continuous extension of the evolution flow of cubic Szegö equation to the whole Hardy class H^2 (which is optimal). The proof relies on spectral theory of Hankel operators on the Hardy class.

A Relation Between the Dirichlet and Regularity Problem for Parabolic PDEs

22 Feb
10:30am

Erik Sätterqvist
University of Edinburgh

In this talk we consider the relationship between the Dirichlet and Regularity problem for parabolic operators of the form

$$L = -div(A\nabla) + \partial_t$$

. Letting $(D_L)_p$ be the statement that the L^p Dirichlet problem is solvable and similarly $(R_L)_p$ for the regularity problem; in the elliptic setting we know from Kenig-Pipher that

$$(R_L)_p => (D_L^*)'_p$$

and the partial converse

$$(D_L^*)'_p + (R_L)_q => (R_L)_p$$

was proved by Shen. The first of these was already shown in the parabolic setting by Dindos-Dyer, where the non-tangential maximal function of the half-time derivative could be ignored. However, after recent progress by Dindos in working with the half-time derivative, we are now able to tackle the second result. This is done by first adapting Shen's approach for $q < p$ and then interpolating L^p against a suitable endpoint space to get the full generality, i.e. that

$$(D_L^*)'_p => (R_L)_p \text{ or } (R_L)_q$$

does not hold for any $1 < q < \infty$. This is joint work with Martin Dindos.

Curvature conditions, Gradient estimates on the Witten Laplacian and Liouville theorems on smooth metric measure spaces

22 Feb
9:30am

Ali Taheri
University of Sussex

In this talk I will present gradient estimates of elliptic types (specifically, of Hamilton and Souplet-Zhang types) for a class of nonlinear parabolic equations on smooth metric measure spaces. The Laplace-Beltrami operator gives its place to the Witten Laplacian and this context the Riemannian metric and potential are taken to evolve with time (a geometric flow). The estimates are established under different curvature conditions and lower bounds on the Bakry-Emery Ricci tensor and are then used to prove a number of important results such as Harnack inequalities, spectral bounds, sharp Logarithmic Sobolev inequalities (LSI) and general Liouville and global constancy results. If time allows, I will present applications of the above to the (super) Perelman-Ricci flow, heat entropy formula and ancient/eternal solutions.
