SQUARE 2 MAGAZINE

Number 10. August 2006.

Welcome to our tenth edtion, complete with photographs from the 2006 graduation. We remind you that news about what you are doing should be sent to the Editor, David Penman, email address dbpenman@essex.ac.uk The magazine will be on our web pages at http://www.essex.ac.uk/maths/SQUARE2 alongside previous editions.

We are lucky, in this edition, to have a piece written for us by two people from outsid the University – Prof. John Dwyer (no stranger to Essex, as he took his Ph.D here in 1981) and his colleague Dr. Hind Zantout.

Departmental News

This year's graduation took place on 21 July. We congratulate all graduates.

Front row (left to right): A guest, Kayley Lamb, Kelly Robinson, Jo Rowsell, Lisa Simpson, Sumita Biswas, Shanaya Dastur and Chang Liu. Back row: Sandra Upton, Carolyn Barry, Mrs. Colchester, Prof. Upton, Dr. Branson, Tom Lofts, Dr. Saker, Xavier Faux, Dr. Harrison, Dr. Penman and Prof. Higgins.

Others graduating this year (but not in this photograph) were: Sharmeen Ali, Richard Ashley, Marie Chesters, Victoria Collings, Shuai Guo, Katie Hall, Spencer Horsey, Natalie Meade and Yilun Song. Some more photographs may be inspected at http://www.essex.ac.uk/maths/misc_pages/graduation2006.htm

Dr. Eleni Maistrelli who was successively an undergraduate, a Ph.D student, and a (much-praised) part-time lecturer in the Department, has now returned to her native Greece to continue her career there. We are very grateful to Eleni for everything she has done for us and wish her all the best for the future.

Four Linked Names in the History of Computing.

Taking a broad view of the history of computing, one could easily trace back to the inventions of the abacus, the development of laws of philosophical logic and the scientific breakthroughs on which today's digital computers are based. Here we briefly describe just four famous contributors whose endeavours have made a lasting impact on the world of computing: **Isaac Newton (1642-1727), Augustus De Morgan (1806-1871), Frank Ramsey (1903-1930) and Adekunle Adeyeye (1970-present).**

Isaac Newton was born into a poor farming family. He was sent to Trinity College, Cambridge, from 1661 to 1696, originally to become a preacher. Instead he studied physics and mathematics (though he himself, in later life, regarded one of his most important works as being a commentary on the Books of Daniel and Revelation). He then spent the rest of his life in London. In 1703, he was elected president of the Royal Society and re-elected each year until his death. When Queen Anne knighted him in 1705, he was the first scientist to be so honoured for his work. Newton developed scientific methods, which were truly universal in scope, and these were described in his *Principia*.

Modern scholarship has not seriously affected his stature in the fields of mathematics, dynamics, celestial mechanics, astronomy, optics, natural philosophy, or cosmology. We now appreciate more fully the extent of his dedication to theology, biblical chronology, prophecy and alchemy. Furthermore, Newton was one of the first to place dynamics on a sound theoretical basis, and from that he derived the *Theory of Statics*. In the Principia, an investigation of physical astronomy was established, as well as the establishment of the law of universal gravitation. Isaac Newton died at the age of 84 in 1727 in London, England. Astonishingly, he never even retired. To the day of his death, he held his position as the president of the Royal Society of London, one of the most prestigious organizations of scientists and mathematicians in England.

We do not always think of Newton as involved in the theory of computing specifically (though the sheer range of his scientific contributions was so vast that they are bound to have affected it). However there is one delightful story, about a letter Newton wrote to Leibniz in 1677. Newton wanted to describe some techniques he had developed as part of his (emerging) “fluxional calculus”, but he didn’t want to divulge the whole thing to Leibniz (who, remember, was developing the calculus himself independently). So, he stopped short in an explanation and instead wrote an anagram (6accdae13eff7i3l9n4o4qrr4s8t12ux, according to http://www.oreillynet.com/onlamp/blog/2004/10/isaac_newton_sha1_and_the_sema.html) which represented his fundamental calculus theory. As someone noted once, this is terribly similar to some of the uses of hash functions in computing today- “I don’t want you to see all my ideas yet, but here is something that can be checked to be the outcome of one of the processes, so that you can see that I did really do it”.

Augustus De Morgan lost the sight of his right eye shortly after birth and it is claimed that he did not excel at school partly because of this physical disability. However, he entered Trinity College, Cambridge in 1823 at the age of just 16. His book *Elements of Arithmetic* (1830) was his second publication. In 1838, he introduced the term *mathematical induction*. The term first appears in De Morgan's article Induction (Mathematics) in the *Penny Cyclopedia.*

He wrote more than 700 articles for the Penny Cyclopedia, which was published by the Society for the Diffusion of Useful Knowledge, set up by the same reformers who founded London University. This society also published another famous work by De Morgan: *The Differential and Integral Calculus*. He recognised the symbolic nature of algebra and the importance of algebras other than ordinary algebra. He introduced De Morgan's laws and probably his greatest academic contribution was as a mathematical logician. It has been recorded in letters that De Morgan corresponded with Charles Babbage and gave private tuition to Ada Lovelace, two other well known pioneers of computing - see

http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Babbage.html

and

http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Lovelace.html

for more on these two.

Frank Ramsey entered Winchester College in 1915 and from there he won a scholarship to Trinity College, Cambridge to study mathematics. At Cambridge, Ramsey became a senior scholar in 1921 and graduated as a Wrangler in the *Mathematical Tripos* of 1923. He then went to Vienna for a short while, returning to Cambridge where he was elected a Fellow of King's College Cambridge in 1924. (He was only the second person ever to be elected to a Fellowship there having not previously studied at the college).

His paper on mathematics *On a Problem of Formal Logic* was read to the London Mathematical Society on 13th December, 1928 and published in the Proceedings of the Society in 1930. This examined methods for determining the consistency of a logical formula, which is clearly of great importance for the theory of computing. Ramsey's paper includes theorems on combinatorics which have led to the study of a whole new area of mathematics called *Ramsey theory*- a related topic was touched upon in Edition 3 of Square2, with the unsolved problem on the function $g(n)$. The celebrated paper of Ramsey has stimulated an enormous study in graph theory and computer science. Most certainly Ramsey theory is now an established and growing branch of combinatorics. In addition, his *theory of utility* is applied widely in the world of both finance and economics. Ramsey suffered an attack of jaundice in 1930 and was taken to Guy's Hospital in London for an operation. He died following the operation at just 27 years of age.

Adekunle Adeyeye was born in Nigeria and he worked in Ibadan as a computer programmer. He left to gain a formal education, earning a degree with first class honours in microelectronics engineering from Cambridge in 1990. He continued his education there, receiving his Ph.D. in 1996 and becoming the first Nigerian to be elected as a junior research fellow at Trinity. At Cambridge, he researched magnetism in thin films, and was able to apply nano-fabrication techniques that enabled the creation of nano-magnets.

Nowadays, Adeyeye is known as a founding researcher at the Information Storage Materials Laboratory at the National University of Singapore. There he works in the field of *spintronics*. Conventional electronics take advantage of the charge of electrons in semi-conducting materials. But electrons also have an important property called *spin*. If Adeyeye succeeds in better utilizing electron spin, he could help revolutionize memory and logic devices, leading to smaller, faster and less power-hungry computers.

So what, apart from a close link with Mathematics, do these four individuals have in common? (Hint: college…)

Some references:

1. http://www.algana.co.uk/FamousNames/FamousNamesFrameset.htm
2. http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Newton.html
3. http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/De_Morgan.html
4. http://www-history.mcs.st-and.ac.uk/Mathematicians/Ramsey.html
5. http://www.ee.nus.edu.sg/ee/view1.asp?user=eleaao

Some photos in this article are courtesy of www.algana.co.uk

Authors:

Prof. John Dwyer earned a Ph.D. in Mathematics from the University of Essex (UK) in 1981 and Dr Hind Zantout earned a Ph.D. in Computing from Kingston University (UK) in 2000. They are members of the Joint Portsmouth-Richmond History of Computing Group. Profiles can be found at:-
http://www.richmond.ac.uk/faculty/dr-john-dwyer.aspx
http://www.richmond.ac.uk/faculty/dr-hind-zantout.aspx

Problems Corner.

Recall our problems from last time. For brevity we only repeat in detail here the two problems for which solutions are available.

1. A mouse is eating a cubic lump of cheese, which consists of 27 small cubes (arranged as in a Rubik's cube: see http://www.thinkgeek.com/images/products/zoom/rubix_cube.jpg if this is not immediately clear to you). Each day, it eats one of the 27 small cubes, and the small cube it eats on day $t+1$ must share a face with the one it ate on day t. The mouse starts in one corner of the big cube. Is it possible for the mouse to eat the small cube which is at the centre of the big cube on the last day?

(Hints: (a) think about the "knight's tour problem in our 5th edition: (b) if you know the words, think about bipartite graphs).

Solution. The point is that we partition the 27 small cubes into two classes. The first class, containing a total of 14 small cubes, contains the 8 vertices at the corners of the big cube and the 6 vertices in the middle of the faces of the big cube. The other class comprises the 13 remaining vertices. (For those with the vocabulary, these are the two vertex classes of the (putative) bipartite graph). We then note (you need to visualise this a bit) that the rule that the small cube eaten on day $t+1$ must share a face with the one it ate on day t implies that the class you are in on day t is the class that you are not in on day $t+1$. (For those with the vocabulary: this says that the two vertex classes above, with two small cubes connected in the graph if and only if they share a face, really is a bipartite graph). Now note that the central small cube (vertex) is in the opposite class to any of the corner small cubes (an easy check: for example it is accessible from the one in the middle of any face, which is clearly in the same class as the four corners of that face). But if we are going to do a walk like this, where we change classes every day, and visit every small cube exactly once starting in the class with 14 cubes/vertices, the other class having only 13 cubes/vertices, then (as we change classes every time) we must also end up in the 14 vertex class. In particular, we cannot end up in the opposite class, at the centre of the cube.

2. Show that every positive integer *n* is a sum of one or more numbers of the form 2^r3^s where *r* and *s* are nonnegative integers and no summand divides another.

[Hint. Induction on *n.* The base step is obvious (?): if *n* is even, what do we know about *n/2* by the induction hypothesis? Otherwise, find the largest power of 3 less than *n*, say 3^m: now consider the number $(n-3^m)/2$. Check the small print holds].

Solution. As hinted, by induction on *n.* The base step *n*=1 is is obvious as $1 = 2^0 3^0$ and with one only summand, there is no other summand to possibly divide it!

Suppose now that the statement is true for all integers less than *n*: we need to show that it holds for *n* as well. If *n* is even, $n/2$ is an integer strictly less than *n*, so by the induction hypothesis $n/2$ is a sum of terms of the form 2^r3^s with no summand dividing any other. Thus, multiplying both sides by 2, we get an expression of *n* as a sum of terms of the form $2^{r+1}3^s$. Further no such term divides any other, as if we did have such a divisibility, say $2^{r+1}3^s$ dividing $2^{m+1}3^n$ then we would have that 2^r3^s divides 2^m3^n in the expression for $n/2$ which is a contradiction.

The other possibility is that *n* is odd. Let then, as in the hint, 3^m be the largest power of less than *n*. Then of course *n*-3^m is even, so $(n-3^m)/2$ exists, and is strictly less than *n*. By the induction hypothesis, $(n-3^m)/2 = \sum 2^{r_i}3^{s_i}$ where no term 2^r3^s divides any other. Thus we get that $n = 3^m + \sum 2^{r_i+1}3^{s_i}$. Again, we cannot have $2^{r+1}3^s$ dividing $2^{m+1}3^n$ as then we would have that 2^r3^s divides 2^m3^n in the expression for $n/2$ which is a contradiction. Thus the only thing that requires a little thought is to check that the term 3^m neither divides nor is divided by any term $2^{r_i+1}3^{s_i}$. Certainly no $2^{r_i+1}3^{s_i}$ divides 3^m as the first number is divisible by 2 but the second isn't. Finally, we do not have 3^m divides $2^{r_i+1}3^{s_i}$ as if we did, we would have that

$$(n-3^m)/2 = \sum 2^{r_i}3^{s_i} > 2^{r_i}3^{s_i} > 2^r3^m$$

using that 3^m divides $2^{r_i+1}3^{s_i}$.

(This is an example where the broad idea of how to do it is not hard to see, given the hints: but did you *really* check everything you have to? Did the proof here?)

And no, since you ask, we are not aware of anyone having got to the bottom of the Navier-Stokes equations since our last edition. The 1,000,000 dollars are still yours for the taking…..

New Problems

As usual, we don't want to know about solutions to the first two, which are (in some sense) standard. Valid solutions to the third problem are a different ball game…

Problem 1. Suppose that T_1 is a triangle with sides of lengths a_1, b_1 and c_1. Similarly let T_2 is a triangle with sides of lengths a_2, b_2 and c_2. Suppose that $a_1 \le a_2, b_1 \le b_2, a_3 \le b_3$ and T_2 has all its angles acute. Does it follow that the area of T_1 is at most that of T_2? [No hints: do it yourselves].

Problem 2. Let A be an $n \times n$ matrix all of whose entries are ±1 and whose rows are mutually orthogonal as vectors (meaning that if, as usual, the ith row is $(a_{i1},....a_{in})$ and the jth is $(b_{i1},....b_{in})$, then we have $\sum_{k=1}^{n} a_{ik} a_{jk} = 0$ whenever $i \neq j$). Such a matrix is called a Hadamard matrix (of order n). For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is clearly a Hadamard matrix. Wait until Problem 3 for more on Hadamard matrices…

The problem for now is: Suppose A has an $a \times b$ submatrix whose entries are all +1. Show that $ab \le n$.

(One possible) hint: Suppose the $a \times b$ submatrix occupies the first a rows and the first b columns. If M is the submatrix occupying the first a rows and last $n-b$ columns (i.e. the rest of the first a rows) what must the matrix MM^T look like? Let $\mathbf{j}$ be an a-vector all of whose entries are 1: can you work out what $MM^T\mathbf{j}$ will look like? Can you get from this statement to a statement about a relevant number being an eigenvalue of MM^T? Now can you see why all these eigenvalues must be positive? (There are many other approaches to the problem).

Problem 3 is also about Hadamard matrices, as defined in Question 2. These might look slightly esoteric objects, but they crop up in all sorts of corners of Mathematics as we shall see. Note that an equivalent condition to A being an $n \times n$ Hadamard matrix is that $AA^T = nI$. (Why?).

Jacques Salamon Hadamard is perhaps better known to mathematicians for proving (at the same time as the Belgian Charles de la Vallée Poussin: they were working independently of each other) that the number of prime numbers less than or equal to n is asymptotically $n/\log(n)$ where log is taken to the base *e*. However he also spent a lot of time on these matrices. For a biography of Hadamard, see http://www-history.mcs.st-andrews.ac.uk/history/Biographies/Hadamard.html

Hadamard himself noted in 1893 that a necessary condition for an an $n \times n$ Hadamard matrix to exist is that $n=1,2$ or a positive multiple of 4. Your (easy) first exercise is to deal with all cases of this statement involving $n \leq 2$. The meat of the assertion is obviously the case of showing that for if $n>2$ we need 4 dividing *n*. You should be able to handle the cases where *n* is odd quite simply (hint: think of a value that the sum of an odd number of ±1s cannot take…). Ruling out values of *n* congruent to 2 modulo 4 is a little harder: the idea is to note that if we multiply some column of a Hadamard matrix by -1, then the resulting matrix is still a Hadamard matrix (why?) and so without loss of generality we can take the first row to be all 1s: then as every other row is orthogonal to this first one, every row must contain an equal number (*r,* say) of 1s and -1s. Now think about the second and third rows (provided there are at least three rows, i.e. provided $n \geq 3$): of the *r* 1s in the second row, if *s* of them are in the same column as a 1 in the third row and *r-s* in the same column as a -1 in the third row, then *t* of the *r* -1s in the second row are in the same place as a +1 in the third row and *r-t* are in the same place as a -1 in the third row. Now we have $s-(r-s)-t+(r-t)=0$ s considering the inner product of second and third row, giving $s=t$. Also the inner product of the third row and the first row gives $s-(r-s)+t-(r-t)=0$ which (as $t=r$) implies that $4s=2r$ so thus $n=2r$ is divisible by 4.

The problem is now to see which of these possible values actually arise. The guess is that the above (not very difficult) necessary condition may well actually be a sufficient condition. No counter example is known.

Problem. Determine whether or not there is an $n \times n$ Hadamard matrix for every *n* divisible by 4.

Again, this is almost certainly very difficult. The conjecture is known to be true for small values of $n=4s$ and the smallest value for which the answer is not at present known is $n=668$. (The construction of an example for $n=428$ was a non-trivial achievement: see http://www.math.ipm.ac.ir/tayfeh-r/papersandpreprints/h**428**.pdf for details: you need background reading beyond this article to understand this paper).

Some of the constructions are a good deal more easy that that. For example, if we have any $m \times n$ matrix *A* and any $p \times q$ respectively, the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix made up of $p \times q$ blocks: the block in the *i*th row of blocks and the *j*th column of blocks is $a_{ij}B$. It is then not hard to show that if *A* and *B* are both Hadamard matrices, so is $A \otimes B$. (Check this yourself?). The upshot of this is that if we have an $m \times m$ Hadamard matrix *A* and any $n \times n$ Hadamard matrix *B*, we have one of order $mn \times mn$.

We can construct a Hadamard matrix of order $q+1$ where q is a power of a prime number, such that $q \equiv 3 \ (4)$ (i.e. 4 exactly divides $q-3$). This construction depends on some properties of finite fields: it is due to R.E.A.C. Paley and hence the resulting Hadamard matrices are called Paley matrices.

The smallest number not covered by these two constructions, together with the existence of Hadamard matrices of orders 1 and 2, is 36 (and you should check that this statement is true!): you can see two ways of getting Hadamard matrices of this order in http://designtheory.org/library/encyc/topics/had.pdf
These two are interesting because they illustrate the links between Hadamard matrices and the theory of designs, which the webpage just mentioned discusses in detail. Hadamard matrices are also used to construct certain kinds of codes, which is (part of) the reason why Prof. Neil Sloane of ATT maintains a library of Hadamard matrices at http://www.research.att.com/~njas/hadamard/index.html There appear to be other links too: e.g. to the theory of integral equations.

We forgot to mention one of the most obvious links of Hadamard matrices with other parts of Mathematics – namely, with geometry. What, one might ask, is the maximum modulus of the determinant of an $n \times n$ matrix A, all of whose entries are ±1? The correct answer turns out to be that the modulus of the determinant in question is at most $n^{n/2}$, with equality if and only if the matrix is Hadamard. Not at all obvious algebraically. However if $x_1, x_2, \ldots x_n$ are the rows of A, then remember the determinant is the volume (n-dimensional) of the parallelpiped whose sides are $x_1, x_2, \ldots x_n$ (a parallelpiped is to n dimensions what a parallelogram is to 2 dimensions: those of you who have done MA114 in recent years have at least seen the case n=2 of this statement, the general case works similarly). Now note that thus

$$|\det(X)| \leq \|x_1\| . \|x_2\| .... \|x_n\|$$

with equality if and only if the vectors $x_1, x_2, \ldots x_n$ are all orthogonal to each other. (Again, if this statement is not obvious to you, think about the case n=2 first). Thus, as we have already seen that such a matrix is Hadamard if and only if the rows are all orthogonal to each other, we deduce the result.

Mathematical Jokes

Suggestions are still made from time to time that certain practitioners of mathematics are less well versed in practical skills than they ought to be. Baa humbug…

A City firm is hiring mathematicians. The three best candidates - a pure mathematician, an applied mathematician, and a graduate in mathematical finance - are asked what starting salary they are expecting.
Pure mathematician: "Can I have £30,000?"
Applied mathematician: "I suppose £60,000 would do”.
Mathematical Finance person: “£300,000 minimum”.
Personnel officer (breathing deeply): "You do realise that we have a graduate in pure mathematics who is willing to do the same work for a tenth of what you are demanding?"
Mathematical Finance person: "Well, I thought of £135,000 for me, £135,000 for you, and £30,000 for the pure mathematician who will do the work…"

A second company is hiring mathematicians, and a pure mathematician, an applied mathematician and a statistician come to the final interviews. The interviewers have heard all about recent dumbing down at British universities so they need to do some checks on the candidates….

Personnel officer: “What is 2+2?”
*Pure mathematician: “*What abelian group are you asking this question in?”

He doesn’t get it. They try again with the applied mathematician.

Personnel officer: “What is 2+2?”
Applied mathematician (*groping for calculator*): “3.9999999.”

He doesn’t get it either. The last candidate is the statistician.

Personnel officer: “What is 2+2?”
Statistician: “What would you like it to be?”
Personnel officer: “Congratulations. You’re hired”.

A third (and, mercifully, last) company is recruiting staff and has included an initiative test in its selection procedure. An engineer, a physicist, and a mathematician are being considered for the post. Each of them is locked in a room for a day - hungry, with a can of food, but without an opener; all they have is pencil and paper. The company’s psychologists wait with bated breath.

At the end of the day, the psychologists first go into the engineer's room. Pencil and paper are unused, but the walls of the room are covered with dents and there is a mess of tinned food all over the floor. The engineer is sitting on the floor: he threw the can against the walls until eventually (on the monkey typing Hamlet principle) it cracked open.

Next they visit the physicist. The paper is covered with formulas, there is one dent in the wall: He calculated how exactly to throw the can against the wall, so that it would crack open. Unfortunately the contents then sprayed around all over the room, so he didn’t get anything to eat either….

Finally they go to see the mathematician. They find this room deeply puzzling since the paper is also full of formulas, the can is still closed, and the mathematician has disappeared. But there are strange noises coming from inside the can.

One of the psychologists gets a tin opener (*well done, sir*) and opens the can. The Mathematician emerges well-fed and unconcerned, but remarking “You really do have to watch your sign slips in this topology lark…”