Marketing New Products in Social Networks.

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Abstract

This article investigates the way in which social interaction affects marketing strategies of a firm looking to introduce a new product. In our model, Word-of-Mouth communication may help consumers to learn their valuation for the new product. We find that demand is generally lower and more elastic when consumers ought to learn their valuations, relative to the case in which consumers know their valuations. The firm, in turn, has incentives to advertise and lower its prices. We provide comparative statics for marketing strategies with respect to changes in the level of social communication and homophily. We find that intensive Word-of-Mouth communication and high levels of social homophily are detrimental for consumer welfare.

1 Introduction.

Occasions in which consumers learn about new products through social interaction are not uncommon. A salient example is given by the social marketing agency BzzAgent\(^1\). Created in 2001, BzzAgent describes itself as the leading social marketing company dedicated to connect people with their favourite brands. In particular, they claim to ”put products in the hands of hundreds of thousands of real consumers and help them share their opinions about them with friends and family via reviews, Facebook posts, photos and videos, blog posts and more”\(^2\). BzzAgent has dedicated lots of its advertising efforts to introduce new products whose value to the consumer is yet to be known. Examples include a campaign made for Penguin Books that introduced a new book called ”The Frog King”, a new hygiene line for Colgate-Palmolive called Pure and Clear, a new line of low calories English bagels for Thomas\(^3\), a new HTC Windows smartphone, between others\(^3\). BzzAgent reports the Pure and Clear campaign reached more than 430k people who didn’t know about the product before and induced more than 98k to try it. The English bagels campaign reached more than 1m people and the HTC campaign reached more than 230k people. Interestingly, BzzAgent does not

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\(^1\)BzzAgent is not alone as a social marketing agency. Another example is found in Tremor, a division of Procter and Gamble, dedicates itself to introduce new products, ”from toothbrushes to TV shows”. See: [24].

\(^2\)See: http://www.bzzagent.com

\(^3\)Ibid.
pay its agents to recommend its products. Agents are given the new product for free and let alone to interact with their peers.

This article studies the way in which advertisers can use social networks to promote new products. Specifically, this paper aims to respond questions as: how does social interaction affect consumer demand for a new product, how should a firm introducing a new product advertise in social networks and how does social interaction affect the profits of the firm, how does it affect welfare? This article develops a framework dedicated to address these questions.

A firm wants to persuade a group of individuals of buying a new product. We assume there is positive correlation between valuations across connected individuals. This incentives the firm to give away free samples of the new product to a subset of individuals. As such, the firm selects the number of free samples to give away and sets a price for potential consumers. Individuals do not know about the existence of the product. Therefore, they do not have a precise idea of how much should they value the new product. Individuals are connected within the population by a social network and may use it to discuss products with their friends. Receivers of the free sample learn their valuation and share it with their friends. Individuals that did not receive a free sample use the information received through social interaction to update their beliefs about their evaluation for the product and make a consumption decision.

This article finds two sets of results. The first set of results considers the way in which the process of learning valuations through social interaction affects consumers’ demand and, therefore, the firm’s marketing strategies. We find that the demand for a new product is generally lower and more elastic when consumers ought to learn their valuation, relative to the case in which they know their valuation for the product. As such, the optimal advertising strategy is larger and optimal pricing strategy lower when consumers have to learn their valuation through social interaction than when they know their valuation for the product.

The second set of results provides comparative statics for marketing strategies along two dimensions of social interaction: the level of social interaction and the level of social homophily. The former refers to the number of people individuals talk to. The latter alludes to extent to which people with similar characteristics are acquainted to each other. Our analysis shows that, when the level of social interaction is low, the firm uses social interaction as a complement to its advertising strategy. However, when the level of social communication is large, the firm is able to substitute advertising with social communication. On the other end, we find that, when the level of social homophily is low, the firm has large incentives to advertise. In contrast, when the level of social homophily is large, the firm is able to substitute advertising with Word-of-Mouth communication. Remarkably, the firm uses pricing as a complement to its marketing strategy. Nevertheless, an increase in the level of social homophily or social interaction incentivizes the firm to increase its price.

As noted earlier, the effect of social interaction on welfare is a central issue for policymakers. Particularly, the explosion of the online marketing industry has opened the debate on regulation. In relation to the latter, the number of free samples the firm distributes.

5For example, Facebook reported profits of $2.5 b USD last year.
our analysis suggest that connectivity and homophily work in favour of the firm. I.e., in social networks in which social interaction and social homophily is large, the firm is able to set higher prices and substitute advertising with social interaction. This contrasts with social networks in which social interaction or social homophily is low, where the opposite holds.

Our paper can be set in-between two different strands in the literature. On the one hand, it is related to the vast body of literature on advertising. On the other hand, our paper is associated with literature on Word-of-Mouth communication and social networks. In this sense, the main contribution of our paper is to provide an understanding of the way in which social communication affects consumer demand and, as a consequence, the design of a marketing strategy when introducing new products.

The present article is closely related to two articles. On the one hand, Galeotti and Goyal (2009) and, on the other, Campbell (2013). Galeotti and Goyal (2009) develop a framework to investigate how can firms and governments use social networks to promote their goals. In particular, they address the following questions: for which product categories are networks important, how should firms use social networks to promote their goals, and how much should a firm be willing to pay to acquire information on social networks? To do so, they develop a model in which there is an agent that chooses a strategy to influence a group of individuals. As in our setup, individual actions are influenced by social interaction. Their interest is centred in understanding how does social networks affect design of optimal social influence strategies. Particularly, the main difference with Galeotti and Goyal (2009) is that our paper provides an understanding of the way in which social communication affects learning consumers’ valuation for new products and how does this process affect demand and marketing strategies.

Campbell (2013), on the other end, develops a framework to study demand, pricing and advertising in presence of social learning via social communication. Campbell (2013) finds that Word-of-Mouth communication is not sufficient for prices to be more elastic and prices to be lower compared to an informed population. As our paper, the author provides comparative statics on results on connectivity. Different from Campbell (2013), we aim to provide a broader understanding of the way in which social homophily affects the learning process and, thus, marketing strategies.

The article is organised as follows. Section 2 presents the model. Section 3 studies consumers’ demand. Section 4 analyses the optimal marketing strategies of the firm. Section 5 assesses welfare. Section 6 provides some extensions to the original model. Section 7 concludes.

2 The Model.

Setup. Consider the problem of a monopolist, $M$, that wants to persuade a unit measure of individuals $\mathcal{N} = [0, 1]$ to buy a new product. Individuals have heterogeneous preferences for the product. In particular, individuals have valuation $v = \{L, H\}$, where $L = 0$ and $H = 1$. Initially, individuals do not have a precise idea of their valuation for the product. However, individuals know their valuation for the product is $v = H$. 

with probability 1/2 and \( v = L \) with the complement. The monopolist cannot distinguish among individuals and their valuation. Individuals and the monopolist know the distribution of preferences.

**Social interaction and learning valuations.** Individuals are initially uninformed about the existence of the product. A fraction \( x \in [0,1] \) becomes aware of the product by receiving a free sample from \( \mathcal{M} \). We refer to \( x \in [0,1] \) as *informed individuals*. Individuals that receive a free sample learn their valuation for the product. The fraction \( (1-x) \) may find out about the product through social interaction. We refer to \( (1-x) \) as *consumers*. We model social interaction as follows. An individual \( i \) that has valuation \( v_i \in \{ H, L \} \), samples \( k \in \mathbb{N}_{+0} \) others. If \( i \) samples informed individual \( j \), then \( j \) truthfully communicates her valuation to \( i \). We assume that if \( i \) samples \( j \) the probability \( v_i = v_j \) is \( h \in [1/2, 1] \). When \( h = 1/2 \), there is random matching of individuals. When \( h = 1 \), individuals with valuation \( H \) are connected only to individuals with valuation \( H \) and individuals with valuation \( L \) are connected only with individuals with valuation \( L \). When \( h \in (1/2, 1) \) there is homophily. Based on the information that \( i \) has received from social interaction, \( i \) updates her beliefs about her valuation of the product. E.g., if \( h = 1/2 \), she does not learn her valuation for the product from social interaction. If \( h = 1 \), she learns perfectly her valuation for the product from social interaction.

**Timeline.** First, the monopolist chooses a fraction \( x \in [0,1] \) of individuals to inform and a price \( p \in [0,1] \) to sell its product. Secondly, informed individuals learn their valuation and social communication takes place. Consumers that interact with informed individuals become informed about the price of the product \( p \) and the valuations of their neighbours \( \{L, H\}^n \). Consumers that do not interact with informed individuals do not learn about the existence of the product. Based on the information received, consumers update their beliefs and make a consumption decision \( s_i : \{p, \{L, H\}^n\} \to \{B, NB\} \), where action \( B \) stands for Buy and action \( NB \) stands for Not Buy. Finally, \( \mathcal{M} \) receives \( \pi = pD(x,p) \), where \( D(x,p) \) is the demand for the product. Consumers that buy the product receive utility \( U(B) = v - p \), consumers that do not buy the product receive utility \( U(NB) = 0 \).

### 3 Demand.

To study the way in which consumers learn their valuation from social interaction, we restrict our analysis to the case in which \( h \in (1/2, 1) \). In order to characterise the demand, we first study the way in which social interaction affects the consumption decision. In this context, social interaction involves sharing individual’s valuation of the product. Particularly, we say informed individual \( j \) *recommends* the product to consumer \( i \), when \( j \) communicates to \( i \) that her valuation for the product is \( v_j = H \). In this way, we define consumer’s \( i \) beliefs about her valuation of the product as follows:

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6When \( h = 1/2 \), consumers do not learn from social interaction. When \( h = 1 \), consumers learn perfectly from social interaction. The latter case can be related to the model of Word-of-Mouth in Galeotti and Goyal (2009).
\[ \mu_i(v_i = H \mid n, r, h) = \frac{h^r(1-h)^{n-r}}{h^r(1-h)^{n-r} + (1-h)^r h^{n-r}} \]

Where \( r \) is the number of recommendations informed consumer \( i \) has received. A consumer will buy the product whenever \( \mu_i(v_i = H \mid n, r, h) \geq p \). Bearing this in mind, it is possible to find the number of recommendations that makes the consumer indifferent between buying or not the product. This is given by:

\[ r^*(n, p, h) = \alpha + \frac{n}{2} \]

Where \( \alpha = \frac{\log\left[\frac{1-x}{p}\right]}{2\log\left[\frac{1-x}{p}\right]} \).

As consumers receive more recommendations, they are more willing to buy the product. In this sense, receiving a number of recommendations above \( r^* \) should imply consumers buy the product. We now proceed to prove this is the case. All proofs are confined to the Appendix.

**Lemma 1.** A consumer that samples \( n > 0 \) informed individuals and receives \( r \) recommendations buys the product at price \( p \) if and only if \( r \geq r^*(n, p, h) \).

Lemma 1 describes the way in which social communication affects the decision of purchase. Using this result, an estimate of the demand function in presence of Word-of-Mouth (WoM hereafter) communication is given by:

\[ D(x, p) = (1-x)x \sum_{n=0}^{k} \left( \begin{array}{c} k \\ n \end{array} \right) x^n (1-x)^{n-k} \sum_{r=r^*}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^{n-r} \]

Where the terms \( \sum_{n=0}^{k} \left( \begin{array}{c} k \\ n \end{array} \right) x^n (1-x)^{n-k} \sum_{r=r^*}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^{n-r} \) represent the probability individuals receive recommendations from their peers. Developing \( D(x, p) \) yields:

\[ D(x, p) = (1-x)x \left( 1-2^{\alpha-1} \left( 1-\left( 1-\frac{1}{\sqrt{2}} \right) x \right)^k \right) \]

Our first result characterises, \( D(x, p) \).

**Lemma 2.** The demand for the product is \( D(x, p) > 0 \) if and only if \( p \in [0, \bar{p}] \) and \( D(x, p) = 0 \) otherwise, where \( \bar{p} = \frac{1}{1+e^{\beta}} \) and \( \beta = \frac{2}{\log[2]} \log \left[ \frac{1-h}{h} \right] \log \left[ 2 \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^{-k} \right] \).

Lemma 2 sets a maximum price for demand to be non-negative. We use the demand in Lemma 2 for all our analysis. The elasticity of the demand is given by:

\[ \epsilon = \frac{2^{\alpha-2} \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^k \log[2]}{(1-p) \left( 1-2^{\alpha-1} \left( 1-\left( 1-\frac{1}{\sqrt{2}} \right) x \right)^k \right) \log \left[ \frac{1-h}{h} \right]} < 0 \]
Different from models that ignore WoM communication, the demand function in our model and, thus, elasticity of the demand, depends on the level of social interaction and the level of homophily between individuals. To contrast our model to one that ignores social interaction, consider the case in which individuals are fully informed about their valuation for the product. In such a case, the demand is given by \( D_{FI} = \frac{1}{2}(1 - x)x \), where FI stands for Full Information. The price elasticity of \( D_{FI} \) is given by \( \epsilon_{FI} = 0 \). As such, demand in presence of WoM communication is more elastic. This particularity suggest that ignoring social interaction may bias estimations of the demand function. In view of this, we now turn to study the way in which social communication impacts demand and discuss how WoM may bias the estimation of the demand.

### 3.1 Changes in the level of social interaction and homophily.

Compared to the case of full information, the demand is more elastic when social interaction is considered. This is due to the fact that two dimensions of social interaction act upon the process of learning the valuation for the product. The level of social interaction, which is related to the probability consumers receive recommendations, and the level of social homophily, which affects the correlation between consumers’ valuation and receiving a recommendation. We begin by analyzing the way in which the level of social interaction affects demand and, in turn, the probability of receiving a recommendation.

**Proposition 1.** Let \( x > 0 \). If \( k \) increases, then:

1. The demand increases.
2. The price elasticity of the demand decreases.

Increasing the level of social interaction increases the flow of information. As a result, the demand for the product increases. Moreover, as consumers learn more about the product, its demand becomes more inelastic. As discussed above, the price elasticity of the demand is related not only to the probability of being informed, but also to the correlation between receiving a recommendation and consumer’s valuation. As such, we, now, study the way in which social homophily affects demand for the product.

**Proposition 2.** Let \( x > 0 \). If \( h \) increases, then:

1. The demand increases.
2. The price elasticity of the demand decreases.

An increase in the level of homophily enhances learning by incrementing the correlation between receiving a recommendation and consumers’ valuation. As a consequence, an increase in the level of homophily increases the demand and reduces its price elasticity. Differences in, both, the level of homophily and level of social interaction affect the estimation of the demand. We now study the implications of ignoring WoM communication on the estimation of the demand.
3.2 Biases in the estimation of the demand.

The explosion of online social networking sites such as Twitter and Facebook\(^7\) has increased the volume of internet commerce. Sites like Amazon and Ebay allow online sellers to connect to social networking platforms to increase their sales. However, the level of social interaction and homophily different communities have on the internet differs between products and social networking platforms. These differences may induce biases in the estimation of the demand for the product. In view of this, we first discuss the way in which the level of social interaction may bias the estimation of the demand. Then, we assess how does social homophily may impact the estimation of the demand.

Consider the case of a monopolist looking to introduce a new product across two groups of individuals. Groups differ in their level of social interaction\(^8\). Group one has level of social interaction \(k\) and group two has level of social interaction \(k'\), where \(k < k'\) respectively. From Proposition 1, it follows that at price \(p\), the monopolist faces a lower demand in group one than in group two. I.e., \(D_k < D_{k'} < D_{FI}\), where \(D_{FI}\) is the level of demand in the situation in which individuals know their valuation (Full Information). As such, ignoring the process of social learning introduces an upward bias on the estimation of the demand. Moreover, even when considering social communication, calculating the demand ignoring differences in the level of social interaction between groups may bias the estimation of the demand.

Now, consider the case in which the monopolist’s aim is to introduce a new product across two groups of individuals. Both groups present the same level of social interaction. However, group one has a level of homophily \(h\) and group two a level of homophily \(h'\), where \(h < h'\). From Proposition 2, it follows that, at price \(p\), the monopolist faces lower demand in group one than in group two. I.e., \(D_h < D_{h'} < D_{FI}\) where \(D_{FI}\) is, once again, the level of social interaction under Full Information. As before, ignoring the process of social learning introduces an upward bias to the estimation of the demand. Furthermore, disregarding social homophily biases the estimation of the demand, even when accounting for the level of social interaction.

Both, ignoring social homophily and different levels of social interaction, introduce biases in the estimation the demand across different groups of individuals. However, the growth of online social connectivity over time raises the issue of the estimation of future demand. This is of particular interest for firms like Facebook and Google whose aim is to introduce their product to new markets. Campbell (2013) relates the WoM bias to the censoring "stock-out" problem in which the true demand for a product is greater than the estimated demand for any given price. In our model, an increase in connectivity and or homophily from period \(t\) to period \(t + 1\) would be reflected in a downward bias of the estimation of the demand due to censoring stock-out. However, unlike the stock-out problem, estimating levels of social homophily or social interaction would allow to correct biases introduced by learning through WoM communication.

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\(^7\)See Introduction.

\(^8\)In social networking applications such as Pair or Path, the level of social interaction is restricted to one and 150 individuals respectively. By design, the level of social interaction on such social networking sites is different among users of Pair and Path. See [23] for more information on such sites.
4 Marketing Strategies.

The fact that connectivity and homophily reduce the price elasticity of the demand allows the monopolist to act strategically to increase its profits. In view of this, we turn to study marketing strategies. I.e., the pair \((x, p)\) comprising amount of advertising and price. Our first set of results compare and contrast marketing strategies for the case in which individuals are aware of their valuation for the product to strategies in a situation where individuals learn their valuation via social interaction. A second set of results studies comparative statics of marketing strategies with respect to the level of social interaction and level of social homophily.

4.1 Contrasting knowing and learning valuations.

Call the situation in which consumers know their valuation for the product as the Full Information case and the situation in which consumers learn their valuation through social interaction as Word-of-Mouth case. Profits for the Full Information case are given by \(\pi_{FI} = p(1 - x)x/2\). Profits for the WoM case are given by:

\[
\pi_k = p(1 - x)x \left(1 - 2^{\alpha - 1} \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)x\right)^k\right)
\]

Where \(\alpha = \log\left[\frac{1-x}{p}\right] / 2\log\left[\frac{1-x}{x}\right]\).

**Proposition 3.** Fix a \(p\). Optimal advertising strategy under Word-of-Mouth communication is equal or greater than under Full Information.

In the model of learning valuations through Word-of-Mouth communication, the monopolist may have additional incentives to inform individuals about its product relative to a monopolist facing a fully informed population. By doing so, the monopolist in the WoM model increases the likelihood individuals learn about its product and engage into consumption. Therefore, the monopolist sets an advertising level equal or larger that would be set to a fully informed population.

**Proposition 4.** Fix an \(x > 0\). Optimal pricing strategy under Word-of-Mouth communication is equal or lower than under Full Information.

Under learning through Word-of-Mouth communication, the demand is more elastic compared to the situation in which consumers are fully informed about their valuations for the product. Therefore the monopolist facing WoM sets a lower price that would be charged to a fully informed population.

4.2 Level of Social Interaction.

Empirical work like the one of Leskovec, Adamic and Huberman (2007) and Keller, Fay and Berry (2007) show that the level of Word-of-Mouth communication is different between groups of products. For instance,
food, dining and media entertainment have the highest level of Word-of-Mouth communication and children
product’s, the lowest. Such variations highlight the importance of studying the way in which different levels
of social interaction affect marketing strategies. Our next set of results considers comparative statics of
marketing strategies with respect to the level of social interaction.

Proposition 5. Fix a $p \in [1/2, \bar{p})$ and let $x_{WoM}^* = \arg\max \pi_k$. There exists a $\bar{k}$ such that if consumers’
degree increases and $k < \bar{k}$, then the monopolist increases its optimal advertising level $x_{WoM}^*$. Otherwise, if
consumers’ degree increases and $k > \bar{k}$, then the monopolist decreases its optimal advertising level $x_{WoM}^*$. More-
over, if consumers’ degree increases, then the profits of the monopolist increase.

When $k < \bar{k}$, few individuals receive advertisements from the firm. As such, an increase in $k$ enlarges the
monopolist’s incentive to advertise. In this sense, the monopolist uses Word-of-Mouth communication as a
complement. In contrast, when $k > \bar{k}$, the monopolist informs many potential buyers. Therefore, an increase
in $k$ decreases the monopolist’s incentives to advertise. In this case, Word-of-Mouth serves as a substitute
to advertising. Finally, an increase in the level of social interaction increases the likelihood consumers hear
about the product. Thus, monopolists’ profits increase.

Proposition 6. Fix an $x > 0$ and let $p_{WoM}^* = \arg\max \pi_k$. If the level of social interaction increases, then
optimal pricing level $p_{WoM}^*$ increases. Moreover, if the level of social interaction increases, profits of the
monopolist increase.

As the level of social interaction increases, consumers are more likely to receive recommendations for
the product. This increases the probability they learn about the product. Therefore, price elasticity of the
demand decreases. As a consequence, the monopolist increases its optimal price $p_{WoM}^*$. Furthermore, for
any advertising level $x > 0$, an increase in the level of social interaction increments the likelihood individuals
learn about the product. Therefore, together with the increase in price, an increase in the level of social
interaction enlarges the profits of the monopolist.

4.3 Social Homophily.

The work of Aral, et al. (2009) uses data from an instant messaging service network to document that ignor-
ing homophily in the estimation of peer-influence in product adoption decisions could lead to a 300-700% bias
in the perceived behavioural contagion. These calculations underscore the importance of studying the way in
which different levels of homophily affect marketing strategies. Our next set of results provide comparative
statics of marketing strategies with respect to the level of homophily.

Proposition 7. Fix a $p \in [1/2, \bar{p})$ and let $x_{WoM}^* = \arg\max \pi_k$. If the level of social homophily increases, then
optimal advertising level $x_{WoM}^*$ decreases. Moreover, if the level of social homophily increases, $\bar{k}$ increases
and the profits of the monopolist increase.
When social homophily increases, the likelihood a consumer purchases the product, upon receiving a recommendation, increases. This changes monopolist’s incentives in different ways. On the one hand, for any level of social interaction, the monopolist is able to substitute advertising with WoM communication. I.e., for any \( k \), if \( h < h' \), then \( x_k' < x_k \). On the other hand, however, an increase in the level of homophily results in more consumers buying the product. As a result, the monopolist has more incentives to advertise. This is reflected with an increment on \( \hat{k} \). Finally, the resulting effect of an increase in the level of social homophily is that profits received by the monopolist increase. We turn to study, now, pricing strategies and the way in which WoM communication affects them.

**Proposition 8.** Fix an \( x > 0 \) and let \( p_{WoM}^* = \arg\max \pi_k \). If the level of social homophily increases, then optimal pricing level \( p_{WoM}^* \) increases. Moreover, if the level of social homophily increases, profits of the monopolist increase.

An increase in the level of social homophily enhances learning by enlarging the correlation between receiving a recommendation and consumers’ valuation. As a consequence, the price elasticity of the demand decreases. This allows the monopolist to increase the optimal price. As a consequence, profits for the monopolist increase too. We now turn to study the way social interaction affects simultaneously, both, pricing and advertising strategies.

### 4.4 Optimal Marketing Strategy.

Finally, we consider the way in which social interaction and homophily affect, both, pricing and advertising strategies.

**Proposition 9.** Let \( (x_{WoM}^*, p_{WoM}^*) = \arg\max \pi \). There exists a \( \hat{k} \) such that if consumers’ degree increases and \( k < \hat{k} \), then the monopolist increases, both, its optimal price \( p_{WoM}^* \) and its optimal advertising level \( x_{WoM}^* \). If consumers’ degree increases and \( k > \hat{k} \), then the monopolist increases its optimal price \( p_{WoM}^* \) while it decreases its optimal advertising level \( x_{WoM}^* \). Moreover, if consumers’ degree increases, then the profits of the monopolist increase.

When the level of social interaction is low, price elasticity of the demand for the product is large. As a result, the monopolist has incentives to stimulate the demand by setting a low price. In contrast, when the level of social interaction is large, demand is price inelastic. Therefore, it is optimal for the monopolist to set higher prices. In this way, pricing strategies act as a complement for the advertising strategy. On the other end, advertising remains the same: when \( k < \hat{k} \), the monopolist uses social communication to complement its advertising strategy. When \( k > \hat{k} \), the monopolist uses social communication as a substitute to its advertising efforts. Lastly, an increase in the level of social interaction increases the probability consumers learn about the product. As a result, profits for the monopolist increase with the level of social interaction.

We ran a series of simulations to explore the impact of an increase in the level of homophily in the optimal advertising strategy of the monopolist. Figure 1 plots the relationship between \( \hat{k} \) and \( h \).
Different from the case in which \( p \) is fixed, in when it is allowed to vary, an increase in homophily reduces \( \hat{k} \).

As before, when homophily increases, informed consumers are more likely to buy the product when receiving a recommendation. However, now that the price is allowed to vary, the monopolist is able to use the price to persuade consumers to buy the product. In this sense, the monopolist can avoid costly advertising through price cuts. This is reflected with a decrease in \( \hat{k} \).

5 Consumer Welfare.

Previous sections underscore the way in which social interaction affects the demand and, as a consequence, marketing strategies and profits of the monopolist. Particularly, we note that an increase in the level of social communication or level of homophily decrease the price elasticity of the demand. As a consequence, changes in marketing strategies result in increased profits for the monopolist. Such variations imply social welfare changes too. We, now, turn to study the way in which social interaction affects social welfare. We begin by providing the following definitions for welfare. Define total Welfare \( W \) as follows:

\[
W = \Pi + CS
\]

Where \( \Pi \) is the producer surplus and \( CS \) stands for consumer surplus. Moreover, define \( CS \) as follows:

\[
CS = \frac{1}{2} D(x, p)U(B \mid v = \{H\}) + \frac{1}{2} D(x, p)U(B \mid v = \{L\})
\]

The following proposition shows the way social welfare changes with social homophily.

**Proposition 10.** If \( h \) increases, then:

1. Total Welfare increases.
2. Consumer Surplus decreases.
Markedly, an increase in the level of social homophily increases Producer’s Surplus, and reduces Consumer Surplus. This comes as a result of the way social homophily affects demand. Incrementing the level of homophily increases the correlation between being informed and consumers valuation. As such, price elasticity of the demand decreases. Such a decrease in the elasticity of the demand is profited by the monopolist which increases the price for the product. As a consequence, profits increase and consumer’s surplus decreases. We, now, turn to study the way in which the level of social interaction affects welfare.

**Proposition 11.** If \( k \) increases, then:

1. **Total Welfare increases.**

2. **Consumer Surplus decreases.**

Enlarging the level of social interaction increases the flow of information and, as a result, decreases the elasticity of the demand. As with social homophily, a decrease in the price elasticity of the demand allows the monopolist to increase the price for the product. Hence, profits increase and consumer surplus decreases.

6 Extensions.

**A new product of unknown quality.** The explosion on the usage of mobile devices in the last year has propitiated thriving markets for web and mobile applications. Examples are the Apple AppStore, Google’s Play store, which offer a wide variety of ”apps”. Particular to the latter is the fact that ex-ante programming effort does not guarantee success\(^9\). As such, app producers are sometimes unaware, ex-ante, of the quality of the product they are producing. We frame our model to convey such cases.

A monopolist \( \mathcal{M} \) wants to persuade \( \mathcal{N} = [0,1] \) to buy a product of type \( \theta = \{H,L\} \). Agents do not know \( \theta \) at its realisation. However, they hold prior \( \Pr[\theta = \{H\}] = 1/2 \). Individuals are homogeneous and have valuation \( v_H = 1 \) for the High-quality product and \( v_L = 0 \) for the Low quality product. Individuals are initially uninformed about the existence of the product. \( \mathcal{M} \) informs a fraction \( x \in [0,1] \). The fraction \((1-x)\) is informed via social interaction. Particularly, individuals interact with \( k \in \mathbb{N}_{+0} \) others\(^10\). Individuals in \((1-x)\) interacting with individuals in \(x\) may find about the product via WoM. The probability that an individual positively recommends a product is \( \Pr[r_j = \{H\} \mid \theta = \{H\}] = h \) and negatively recommends \( \Pr[r_j = \{L\} \mid \theta = \{H\}] = h \). We refer to \( h \in [1/2,1] \) as the informativeness level of recommendations. The model develops as follows \( \mathcal{M} \) chooses \( x \) and \( p \in [0,1] \). Individuals in \( x \) engage into social interaction. Consumers that receive recommendations observe the number of positive and negative recommendations they receive and make a consumption decision. A consumer that buys the product receives utility \( U(\{B\} \mid \theta = \{H\}) = 1 - p \) if she buys the high quality product, \( U(\{B\} \mid \theta = \{L\}) = -p \) if she buys

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\(^9\)As an example, consider ”Snapchat”, a photo sharing mobile application whose ”ex-ante” quality cannot be judged. See: [5]

\(^10\)See Section 2 for a description of social interaction
the low quality product and \( U(\{NB\}) = 0 \) if she does not buy the product.

From results in Proposition 1-9, we learn that, when the level of social interaction increases, price elasticity of the demand increases too. As a result optimal price level increases and the optimal advertising level first increases and then decreases. Moreover, an increase in the informativeness level of recommendations has as a consequence that \( M \) is able to substitute to a greater extent advertising with WoM communication.

**Targeted strategies.** Now a days, the amount of information on consumers’ level of social interaction that can be accessed through technology is very large\(^\text{11}\). This, together with the results found in previous sections, calls the attention to the opportunity \( M \) has to tailor their advertising strategy according to individuals’ level of social interaction. We now develop a model in which the monopolist knows the fraction of consumers that sample \( k \) individuals and derive the monopolist’s optimal advertising strategies.

Suppose that the monopolist is able to partition \( N \) into \( K \) groups. Group \( k \in \{1, 2, \ldots, K\} = O \) contains the fraction \( P(k) \) of consumers that sample \( k \) individuals. Assume the monopolist knows \( P \). The strategies of the monopolist consist on an advertising vector \( x = (x_1, x_2, \ldots, x_K) \), with \( x_k \in [0, 1] \). The expected net profits of a monopolist using strategies \( x \) are given by:

\[
\Pi(x \mid P) = \sum_k P(k)p(1 - x_k)\Theta(x)
\]

Where:

\[
\Theta(x) = \sum_l P(l)x_l \left(1 - 2^{\gamma - 1} \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)x_l\right)^l\right)
\]

Where \( \gamma = \frac{\log\left[\frac{1-x}{2}\right]}{2\log\left[\frac{1}{\sqrt{2}}\right]} \) and \( \Theta(x) \) is the vector of probabilities of receiving \( r \geq r^* \) recommendations, given a consumer in \( k \) is consulting an individual in group \( l \in O \).

We now proceed to describe the way in which the strategies of the monopolist change between groups. To do so, we introduce the following definition. For a given subset \( O' \subseteq O \), we say the advertising strategy is *increasing* if \( x_{k+1} \geq x_k \) for every \( k \in O' \). Analogously, we say that an advertising strategy is *decreasing* if \( x_{k+1} \leq x_k \) for every \( k \in O' \).

**Proposition 12.** The optimal strategy of the monopolist can be either increasing in degree or, first increasing in degree and then decreasing in degree.

Proposition 12 shows that when \( M \) has information on consumers’ level of social interaction, optimal advertising strategies across groups may differ. On this matters, we recognise two cases. For groups in

\[^{11}\text{As an example, see http://www.socialbakers.com, a webpage dedicated to sell detailed information on the level of social interaction in different countries. The webpage includes information on social networking sites such as Facebook, Google+, YouTube, Twitter, etc.}\]
which the likelihood consumers receive positive recommendations is low, \( M \) uses social communication to complement its advertising efforts. On the other hand, for group in which the likelihood consumers receive recommendations is high, \( M \) substitutes its advertising efforts with social communication.

Different approaches at the time of implementing Word-of-Mouth marketing campaigns accentuate the importance on studying how variations in the dispersion level of social interaction affect the optimal strategies a monopolist should follow \(^\text{12}\). For that reason, we now turn to study the effects of changes in the degree distribution on profits. In particular, we are interested in studying how changes in dispersion of the level of social interaction affect \( M \)’s profits. We refer to a Mean Preserving Spread (MPS) of distributions as an increase in dispersion in the level of social interaction. Put formally, let \( P \) and \( P' \) be different degree distributions defined on \( k \in O \). Given \( P \), let \( \mathcal{P} \) be its cumulative degree distribution, so that \( \mathcal{P} : \{0, 1, 2, ..., K\} \to [0, 1] \)

\[
\mathcal{P}(y) = \sum_{k=0}^{y} P(k).
\]

**Definition MPS.** \( P' \) is a Mean Preserving Spread (MPS) of \( P \) if and only if \( P \) and \( P' \) have the same mean and \( \sum_{k=0}^{y} \mathcal{P}(k) \leq \sum_{k=0}^{y} \mathcal{P}'(k) \) for every \( y \in \{0, 1, 2, ..., K\} \).

We now proceed to study the effects of a MPS on profits.

**Proposition 13.** Let \( P' \) be a Mean Preserving Spread of \( P \). Then, optimal profits under \( P' \) are lower than under \( P \).

A MPS increases, both, the number of individuals with low and high level of social interaction. Given the fact that the value of receiving an additional positive recommendation is increasing, but concave in the overall number of recommendations, an increase in dispersion reduces the potential profits of the monopolist.

### 7 Discussion and conclusions.

We developed a framework to understand the way in which social interaction affects marketing strategies of a firm looking to introduce a new product. In our paper, social interaction involves sharing consumers valuation of the new product. Our study provides insights on the way in which consumers learn their valuation of the product through Word-of-Mouth communication. Particularly, we find that the demand is generally lower and more elastic when consumers ought to learn their valuation, relative to the case in which they know their valuation for the product. However, the demand and its price elasticity are dependent on the amount of information consumers receive via WoM communication, and the extent to which consumers learn from social interaction.

\(^\text{12}\) An example that underlines this can be found on BzzAgent. BzzAgent is a firm operating in the U.S., U.K. and Canada, and is dedicated to engineer and implement different types of Word-of-Mouth campaigns using social media. Examples range from the campaign BzzAgent organised for Colgate-Palmolive, in which free samples and discounts were distributed among BzzAgent’s influencers, to the campaign the agency arranged for TrendMicro, in which BzzAgent used social media platform YouTube to increase awareness of the importance of internet security among children. For more cases see: [http://www.bzzagent.com](http://www.bzzagent.com)
The level of social interaction is related to the amount of information consumers receive. When WoM is sparse, consumers aren’t able to learn much about their valuation for the product. As a consequence, relative to the situation in which consumers know their valuation, demand is lower and its price elasticity larger. This implies the firm uses its marketing strategy to complement its sale efforts. It sets a low price and advertises more. In contrast, when WoM is intense, consumers are more likely to learn their valuation from social interaction. Demand is larger and its elasticity lower compared to the case in which WoM is sparse. As such, the firm is able to substitute advertising with WoM and set a higher price.

The level of social homophily is related to the extent to which consumers learn from social interaction. When the level of social homophily is low, recommendations do not help consumers learn their valuation. Hence, demand is low and its price elasticity large when compared to the case in which consumers know their valuation. As a result, the firm complements its sale efforts by setting a low price and advertising extensively. However, when the level of social homophily is large, recommendations are useful for the consumer to learn her valuation. Therefore the results reverse.

Interestingly, results above put forward the possibility that intensive WoM communication and or a high level of social homophily is detrimental to welfare and consumer surplus. Our study suggests policymakers could restrict advertising that uses information on consumers’ interests and demographics, as they provide information on the level of homophily of individuals.

A Proofs.

**Lemma 1.** A consumer that samples \( n > 0 \) informed individuals and receives \( r \) recommendations buys the product at price \( p \) if and only if \( r \geq r^*(n, p, h) \).

**Proof.** Let \( r^*(n, p, h) \) be the number of recommendations that yield \( \mu(v_i = \{H\} \mid n, r, h) = p \). As \( h \in (1/2, 1) \), then:

\[
\frac{d\mu}{dr} = -\frac{2((1-h)h)^{2r+n}(\log(1-h) - \log(h))}{((1-h)^n h^{2r} + (1-h)^{2r} h^n)^2} > 0
\]

Therefore, it has to be the case that if \( r' > r^*(n, p, h) \), then \( \mu(v_i = \{H\} \mid n, r', h) - p > 0 \).

**Lemma 2.** The demand for the product is \( D(x, p) > 0 \) if and only if \( p \in [0, \bar{p}] \) and \( D(x, p) = 0 \) otherwise, where \( \bar{p} = \frac{1}{1+c^2} \) and \( \beta = \frac{2}{\log[2]} \log \left[ \frac{1-h}{h} \right] \log \left[ 2 \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^{-k} \right] \).

**Proof.** We begin by proving \( D(x, p) \) is decreasing in \( p \). Then, we define the value \( \bar{p} \) as the price that makes \( D(x, \bar{p}) = 0 \). Finally, we find \( \bar{p} \). Taking \( \frac{dD}{dp} \) yields:

\[
\frac{dD}{dp} = \frac{2^{\alpha-2}(1-x)x \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^k \log[2]}{(1-p)p \log \left[ \frac{1-h}{h} \right]}
\]
Where $\alpha = \frac{\log\left[\frac{1-x}{x}\right]}{2\log\left[\frac{1-p}{p}\right]}$. As $x \in [0, 1]$, $p \in [0, 1]$ and $h \in (1/2, 1)$, then $\frac{dD}{dp} \leq 0$. Define $\bar{p}$ as the value of $p \in [0, 1]$ such that $D(x, \bar{p}) = 0$. Equating $D(x, \bar{p}) = 0$ and solving for $\bar{p}$ yields:

$$\bar{p} = \frac{1}{1 + e^\beta}$$

Where $\beta = \frac{2}{\log[2]} \log\left[\frac{1-h}{h}\right] \log\left\{2 \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)x\right)^{-k}\right\}$. Finally, with $\frac{dD}{dp}$ and $\bar{p}$ we know $D(x, p) > 0$ for $p \in [0, \bar{p})$ and $D(x, p) = 0$ otherwise.

**Proposition 1.** Let $x > 0$. If $k$ increases, then:

1. The demand increases.
2. The price elasticity of the demand decreases.

**Proof.** To prove (1), we take the derivative of $D$ with respect to $k$:

$$\frac{\partial D}{\partial k} = -2^{a-1}(1-x)xA^k \log[A]$$

Where,

$$\alpha = \frac{\log\left[\frac{1-p}{p}\right]}{2\log\left[\frac{1-h}{h}\right]}$$

and,

$$A = \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)x\right)$$

As $x \in (0, 1)$, then $0 < A \leq 1$ and $\log[A] < 0$. Therefore $\frac{dD}{dk} > 0$, which proves (1).

To prove (2), construct the price elasticity of the demand:

$$\epsilon = -\frac{1}{1 - p} \left(\frac{2^{a-2}A^k \log[2]}{(1 - 2^{a-1}A^k) \log\left[\frac{1-h}{h}\right]}\right)$$

Where $\alpha$ and $A$ are as above. Now, take the derivative of $\epsilon$ with respect to $k$:

$$\frac{\partial \epsilon}{\partial k} = -\frac{2^{a-2}A^k \log[2] \log[A]}{(1 - 2^{a-1}A^k)(1-p) \log\left[\frac{1-h}{h}\right]} \left(1 + \frac{A^k}{2(1 - 2^{a-1}A^k)}\right)$$

As $x \in (0, 1)$, then $\log[A] < 0$. Moreover, $p \in [0, \bar{p})$ and $h \in (1/2, 1)$. Therefore $\frac{d\epsilon}{dk} < 0$. Therefore $\frac{d\epsilon}{dk} < 0$.

**Proposition 2.** Let $x > 0$. Moreover, $p \in [0, \bar{p})$ and $h \in (1/2, 1)$. Therefore $\frac{d\epsilon}{dk} < 0$.

1. The demand increases.
2. The price elasticity of the demand decreases.
Proof. To prove (1), we take the partial derivative of $D(x, p)$ with respect to $h$. This yields:

$$ \frac{\partial D}{\partial h} = -\alpha 2^{\alpha - 1} (1 - x) x A^k \log[2] (1 - h) h \log \left[ \frac{1 - h}{k} \right] > 0 $$

Where $\alpha = \frac{\log[\frac{1 - x}{x}]}{2 \log[\frac{1 - x}{x}]}$ and $A = \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)$. As $x \in (0, 1]$ and $h \in (1/2, 1)$, $\frac{\partial D}{\partial h} > 0$. To prove (2), we take the derivative of $\epsilon$ with respect to $h$. This yields:

$$ \frac{\partial \epsilon}{\partial h} = \frac{2^{\alpha - 3} A^k \log[2] \gamma}{(1 - h) h (1 - p) (1 - 2^{\alpha - 1} A^k) \log[2] \left[ \frac{1 - h}{k} \right]^3} > 0 $$

Where $\alpha$ and $A$ are as above, $h \in (1/2, 1)$ and $\gamma = \left( 2 \left( 1 - 2^{\alpha - 1} A^k \log \left[ \frac{1 - h}{k} \right] + \log[2] \log \left[ \frac{1 - x}{p} \right] \right) \right) < 0$. Then $\frac{\partial \epsilon}{\partial h} > 0$.

**Proposition 3.** Fix a $p$. Optimal advertising strategy under Word-of-Mouth communication is equal or larger than under Full Information.

Proof. Taking the FOC for $\pi_{FI}$ and solving for $x_{FI}^*$ yields $x_{FI}^* = 1/2$. Now, taking the derivative of $\pi_k$ with respect to $x$, equating the results to zero yields:

$$ \frac{\partial \pi_k}{\partial x} = 2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right) k (1 - x_{WoM}^*) x_{WoM}^* A^{k - 1} + (1 - 2 x_{WoM}^*) (1 - 2^{\alpha - 1} A^k) = 0 $$

Where $\alpha = \frac{\log[\frac{1 - x}{x}]}{2 \log[\frac{1 - x}{x}]}$ and $A = \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x_{WoM}^* \right)$.

Therefore:

$$ 2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right) k (1 - x_{WoM}^*) x_{WoM}^* A^{k - 1} = -(1 - 2 x_{WoM}^*) (1 - 2^{\alpha - 1} A^k) $$

As $x \in [0, 1], k \geq 0$, and $h \in (1/2, 1)$, the LHS of equation (5) is non-negative. For the RHS to be non-negative too $x_{WoM}^* \in [1/2, 1]$. As such, $x_{WoM}^* \geq x_{FI}^*$.

**Proposition 4.** Fix an $x > 0$. Optimal pricing strategy under Word-of-Mouth communication is equal or lower than under Full Information.

Proof. Let $\pi_{FI} = p(1 - x) x/2$ be the profits of a monopolist facing a population that know their valuation for the product. In such a case, profit maximising price for the monopolist is $p_{FI}^* = 1$. On the other end, it follows from Definition 1 that the price a monopolist under WoM sets is $p_{WoM}^* \leq p_{FI}^*$.

**Proposition 5.** Fix a $p \in (1/2, \bar{p})$ and let $x_{WoM}^* = \text{argmax} \pi_k$. There exists a $\bar{k}$ such that if consumers’ degree increases and $k < \bar{k}$, then the monopolist increases its optimal advertising level $x_{WoM}^*$. Otherwise, if consumers’ degree increases and $k > \bar{k}$, then the monopolist decreases its optimal advertising level $x_{WoM}^*$. Moreover, if consumers’ degree increases, then the profits of the monopolist increase.
Proof. We use the Implicit Function Theorem to calculate \( \frac{dx^\ast_{WoM}}{dk} \).

Calculating \( \frac{\partial^2 \pi_k}{\partial x^2_{WoM}} \) yields:

\[
\frac{\partial^2 \pi_k}{\partial x^2_{WoM}} = -2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right)^2 (k - 1) kp(1 - x^\ast_{WoM}) x^\ast_{WoM} (A^\ast)^{k - 2} + 2^{\alpha} \left( 1 - \frac{1}{\sqrt{2}} \right) kp(1 - 2x^\ast_{WoM}) (A^\ast)^{k - 1} - 2p(1 - 2^{\alpha - 1} (A^\ast)^k)
\]

Where:

\[
A^\ast = \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x^\ast_{WoM} \right)
\]

And \( \alpha = \frac{\log \left( \frac{1 - x}{1 - A^\ast} \right)}{2 \log \left( \frac{1 - x}{1 - A^\ast} \right)} \). By Proposition 3, we know that \( x^\ast_{WoM} \in [1/2, 1] \). Then \( \frac{\partial^2 \pi_k}{\partial x_{WoM}} \leq 0 \).

Now, calculating \( \frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} \) yields:

\[
\frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} = 2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right) p(1 - x^\ast_{WoM}) x^\ast_{WoM} (A^\ast)^{k - 1} (1 + k \log [A^\ast]) - 2^{\alpha - 1} p(1 - 2x^\ast_{WoM}) (A^\ast)^k \log [A^\ast]
\]

Where \( A^\ast \) and \( \alpha \) are as above. The derivative \( \frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} \) does not provide much information so as to determine its sign. Therefore, we use the First Order Condition to simplify \( \frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} \). Simplifying:

\[
\frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} = -p(1 - 2x^\ast_{WoM}) \kappa(k, x^\ast_{WoM})
\]

Where:

\[
\kappa(k, x^\ast_{WoM}) = \frac{1}{k} \left( 1 - 2^{\alpha - 1} (A^\ast)^k \right) + \log [A^\ast]
\]

By Proposition 3, we know the sign of \( \frac{\partial^2 \pi_k}{\partial x_{WoM} \partial k} \) depends on \( \kappa(k, x^\ast_{WoM}) \). We now proceed to show that \( \kappa \) is decreasing on \( k \) and that there exists a \( \bar{k} \) such that \( \kappa(k, x^\ast_{WoM}) = 0 \).

Let \( k = 1 \). Then \( \kappa(1, x^\ast_{WoM}) \) is given by:

\[
\kappa(1, x^\ast_{WoM}) = (1 - 2^{\alpha - 1} A^\ast) + \log [A^\ast]
\]

By Proposition 3, \( x^\ast_{WoM} \in [1/2, 1] \), then \( \kappa(1, x^\ast_{WoM}) > 0 \). Now, let \( k = K \). Then \( \kappa(K, x^\ast_{WoM}) \) is given by:
\[ \kappa(K, x_{WoM}^*) = \frac{1}{K} (1 - 2^{\alpha-1} A^*) + \log[A^*] \]

When \( k = K \), the value of \( \kappa(K, x_{WoM}^*) \) is dependent on \( K \) and \( x_{WoM}^* \). Notice, however, that as \( K \to \infty \), then \( \kappa(K, x_{WoM}^*) \to \log[A^*] \). By Proposition 3, \( x_{WoM}^* \in [1/2, 1] \). Therefore \( \log[A^*] < 0 \). This implies that when as \( K \to \infty \), then \( \kappa(K, x_{WoM}^*) < 0 \) for any \( x_{WoM}^* \in [1/2, 1] \). We now proceed to show that \( \frac{dk}{dx} \leq 0 \).

Taking the derivative of \( \kappa \) with respect to \( k \) yields:

\[
\frac{dk}{dk} = -\frac{1}{2k^2} \left[ 2 - 2^{\alpha} A^{k-1} \left( A^* (1 - k \log[A^*]) - k^2 A^* x_{WoM}^* \right) \right]
\]

As \( 0 < A^* \leq 1 \), then \( \frac{dk}{dx} \leq 0 \). This, together with \( \kappa(1, x_{WoM}^*) > 0 \) for any \( x_{WoM}^* \in [1/2, 1] \) and \( \kappa(K, x_{WoM}^*) < 0 \) for any \( x_{WoM}^* \in [1/2, 1] \) for a sufficient large \( K \), implies that there exists a \( \bar{k} \) such that \( \kappa(k, x_{WoM}^*) = 0 \). Moreover, the latter conveys that if \( k < \bar{k} \), then \( \kappa > 0 \). Otherwise, if \( k > \bar{k} \), then \( \kappa < 0 \).

Finally, using the IFT if \( k < \bar{k} \) yields \( \frac{d^2 x_{WoM}}{dk} > 0 \). If \( k > \bar{k} \), yields \( \frac{d^2 x_{WoM}}{dk} < 0 \).

Proposition 6. Fix an \( x > 0 \) and let \( p_{WoM}^* = \arg\max_k \pi_k \). If the level of social interaction increases, then optimal pricing level \( p_{WoM}^* \) increases. Moreover, if the level of social interaction increases, profits of the monopolist increase.

Proof. In Step 1, we use the Implicit Function Theorem to prove that \( \frac{dp_{WoM}^*}{dk} > 0 \). In Step 2, we use the result in Step 1 to prove that \( \frac{d\pi_k}{dk} > 0 \).

Step 1. Taking the second derivative of \( \pi_k \) with respect to \( p_{WoM}^* \) yields:

\[
\frac{\partial^2 \pi_k}{\partial p_{WoM}^2} = -\frac{2^{\alpha-3}(1-x)x A^k \log[2]}{(1 - p_{WoM}^*)^2 p_{WoM}^* \log \left[ \frac{1-\alpha}{h} \right]^2}
\]

\[
+ \frac{2^{\alpha-1}(1-x)x A^k \log[2]}{(1 - p_{WoM}^*) p_{WoM}^* \log \left[ \frac{1-\alpha}{h} \right]}
\]

Where \( \alpha = \frac{\log \left[ \frac{1-\alpha}{h} \right]}{2 \log[1-\alpha]} \) and \( A = \left( 1 - \left( 1 - \frac{x}{\sqrt{2}} \right) x \right) \). As \( h \in (1/2, 1) \) and \( x > 0 \), then \( \frac{\partial^2 \pi_k}{\partial p_{WoM}^2} \leq 0 \). Now, taking the derivative of \( \pi_k \) with respect to \( k \) and with respect to \( p \) yields:

\[
\frac{\partial^2 \pi_k}{\partial k \partial p} = -2^{\alpha-1}(1-x)x A^k \left( 1 - \frac{\log[2]}{2(1 - p_{WoM}^*) \log \left[ \frac{1-\alpha}{h} \right]} \right) \log[A]
\]

As \( x \in [0,1] \) and \( h \in (1/2, 1) \), then \( \frac{\partial^2 \pi_k}{\partial k \partial p} \geq 0 \). From Step 1, we know that \( \frac{\partial^2 \pi_k}{\partial p_{WoM}^*} \). Therefore \( \frac{dp_{WoM}^*}{dk} > 0 \).

Step 2. We take, now, the derivative of \( \pi_k \) with respect to \( k \) at \( p_{WoM}^* \). This yields:
\[
\frac{d\pi_k}{dk} = -2^{\alpha-1} p_{WoM}^*(1-x)xA^k \log[A]
\]

Where \(\alpha\) and \(A\) are as in Step 1. As \(x \in (0,1]\), then \(\frac{d\pi_k}{dk} > 0\).

**Proposition 7.** Fix a \(p \in [1/2, \bar{p}]\) and let \(x_{WoM}^* = \arg\max \pi_k\). If the level of social homophily increases, then optimal advertising level \(x_{WoM}^*\) decreases. Moreover, if the level of social homophily increases, \(\bar{k}\) increases and the profits of the monopolist increase.

**Proof.** In Step 1, we use the Implicit Function Theorem to calculate \(\frac{dx_{WoM}^*}{dh}\). In Step 2, we make use of this result to calculate \(\frac{d\bar{k}}{dh}\). In Step 3, we calculate \(\frac{d\pi_k}{dh}\).

**Step 1.** From the proof of Proposition 5, we know that \(\frac{\partial^2 \pi_k}{\partial x_{WoM} \partial h} \leq 0\). We, now, calculate \(\frac{\partial^2 \pi_k}{\partial x \partial h}\).

\[
\frac{\partial^2 \pi_k}{\partial x \partial h} = -\left(1 - \frac{1}{\sqrt{2}}\right)ck(1-x_{WoM}^*) x_{WoM}^* (A^*)^{-1} + c(1-2x_{WoM}^*) (A^*)^k
\]

Where \(\alpha = \frac{\log[\frac{1-x}{1-2}]}{2\log[\frac{1-x}{1-2}]}\), \(A^* = \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right)x_{WoM}^*\right)\) and \(c\) stands for:

\[
c = -\frac{\alpha 2^{\alpha-1} \log[2]}{(1-h)h \log \left[\frac{1-h}{h}\right]} \geq 0
\]

From Proposition 3, we know that \(x_{WoM}^* \in [1/2,1]\). Moreover, as \(h \in (1/2,1)\), then \(\frac{\partial^2 \pi_k}{\partial x \partial h} \leq 0\). Using the Implicit Function Theorem, we conclude \(\frac{dx_{WoM}^*}{dh} \leq 0\).

**Step 2.** To prove that as \(h\) increases, \(\bar{k}\) increases, we proceed to calculate \(\frac{dk}{dh}\) using \(\kappa = 0\) as an implicit function. Taking the derivative of \(\kappa\) with respect to \(h\) yields:

\[
\frac{\partial \kappa}{\partial h} = -\frac{\alpha}{\bar{k}} \left(2^{\alpha-1} (A^*)^{k} \log[2] \right) + \frac{1}{A^*} \left(1 - \frac{1}{\sqrt{2}}\right) \frac{dx_{WoM}^*}{dh}
\]

Where \(\alpha\) and \(A^*\) are as in Step 1.

By Proposition 3, we know that \(x_{WoM}^* \in [1/2,1]\). Moreover, \(p \in [1/2, \bar{p}]\) and \(h \in (1/2,1)\). Therefore \(\frac{\partial \kappa}{\partial h} > 0\). Now, take the derivative of the implicit function with respect to \(\bar{k}\):

\[
\frac{\partial \kappa}{\partial \bar{k}} = -\frac{1}{\bar{k}} \left(1 - 2^{\alpha-1} (A^*)^k\right) + \frac{1}{\bar{k} \frac{dk}{dh}} \left(1 - 2^{\alpha-1} (A^*)^k\right)
\]

Where \(\alpha\) and \(A^*\) are as in Step 1. By Proposition 3, \(x_{WoM}^* \in [1/2,1]\), then \(\frac{\partial \kappa}{\partial \bar{k}} < 0\). This yields \(\frac{\partial \kappa}{\partial h} > 0\) and \(\frac{\partial \kappa}{\partial \bar{k}} < 0\). Hence, it has to be the case that \(\frac{dk}{dh} > 0\).

**Step 3.** We calculate, now, \(\frac{d\pi_k}{dh}\) at \(x_{WoM}^*\).
\[
\frac{d\pi_k}{dh} = -\frac{\alpha 2^{\alpha - 1} p \log[2](1 - x^*_WOM)x^*_WOM (A^*)^k}{(1 - h)h \log \left[ \frac{1-h}{h} \right]}
\]

Where \(\alpha\) and \(A^*\) are as in Step 1. From Proposition 3, we know that \(x^*_WOM \in [1/2, 1]\). Moreover, as \(h \in (1/2, 1)\), then \(\frac{d\pi_k}{dh} \geq 0\).

**Proposition 8.** Fix an \(x > 0\) and let \(p^*_WOM = \arg\max \pi_k\). If the level of social homophily increases, then optimal pricing level \(p^*_WOM\) increases. Moreover, if the level of social homophily increases, profits of the monopolist increase.

**Proof.** In Step 1, we use the Implicit Function Theorem to calculate \(\frac{dp^*_WOM}{dh}\). In Step 2, we calculate \(\frac{d\pi_k}{dh}\).

**Step 1.** From Step 1 in Proposition 6, we know that \(\frac{\partial^2 \pi_k}{\partial p^*_WOM} \leq 0\). We, now, calculate \(\frac{\partial^2 \pi_k}{\partial h \partial p^*_WOM}\):

\[
\frac{\partial^2 \pi_k}{\partial h \partial p^*_WOM} = -c \frac{2^{\alpha - 2}}{(1 - p^*_WOM)p^*_WOM} \left( 1 + \frac{\alpha \log[2]}{p^*_WOM} \right)
\]

Where \(\alpha = \frac{\log \left[ \frac{1-p^*_WOM}{p^*_WOM} \right]}{2 \log \left[ \frac{1-h}{h} \right]}\) and \(c\) stands for:

\[
c = -\frac{(1 - x) \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^k \log[2]}{(1 - h)h \log \left[ \frac{1-h}{h} \right]^2}
\]

As \(x \in (0, 1)\) and \(h \in (1/2, 1)\), then \(\frac{\partial^2 \pi_k}{\partial h \partial p^*_WOM} \geq 0\).

**Step 2.** We calculate the derivative of \(\pi_k\) with respect to \(h\) at \(p^*_WOM\):

\[
\frac{d\pi_k}{dp^*_WOM} = -\frac{\alpha 2^{\alpha - 1} p^*_WOM (1 - x) \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right) x \right)^k \log[2]}{(1 - h)h \log \left[ \frac{1-h}{h} \right]}
\]

As \(x \in [0, 1]\) and \(h \in (1/2, 1)\), then \(\frac{d\pi_k}{dp^*_WOM} \geq 0\).

**Proposition 9.** Let \((x^*_WOM, p^*_WOM) = \arg\max \pi\). There exists a \(\hat{k}\) such that if consumers’ degree increases and \(k < \hat{k}\), then the monopolist increases, both, its optimal price \(p^*_WOM\) and its optimal advertising level \(x^*_WOM\). If consumers’ degree increases and \(k > \hat{k}\), then the monopolist increases its optimal price \(p^*_WOM\) while it decreases its optimal advertising level \(x^*_WOM\). Moreover, if consumers’ degree increases, then the profits of the monopolist increase.

**Proof.** This proof of will proceed in four steps. First, we will calculate the price level \(p^*_WOM\) that solves \(\frac{\partial \pi_k}{\partial x} = 0\) and \(\frac{\partial \pi_k}{\partial p} = 0\) simultaneously. Second, we will use \(p^*_WOM\) and the IFT to calculate the sign of the derivative \(\frac{dx^*_WOM}{dk}\). Third, we will use the envelope theorem and \(\frac{dx^*_WOM}{dk}\) to calculate \(\frac{dp^*_WOM}{dk}\). Finally, we will prove the result on profits.
Step 1. Taking the derivative of $\pi_k$ with respect to $x$ and equating it to zero yields:

$$\frac{\partial \pi_k}{\partial x} = 2^{\alpha^*-1} \left(1 - \frac{1}{\sqrt{2}}\right) kp_{WoM}(1 - x_{WoM}^*) x_{WoM}^*(A^*)^{k-1} + p_{WoM}^*(1 - 2x_{WoM}^*) \left(1 - 2^{\alpha^*-1}(A^*)^k\right) = 0$$

Where:

$$\alpha^* = \frac{\log \left[\frac{1 - p_{WoM}^*}{p_{WoM}}\right]}{2\log \left[\frac{1}{1-h}\right]}$$

And $A^* = \left(1 - \left(1 - \frac{1}{\sqrt{2}}\right) x_{WoM}^*\right)$. Now, taking the derivative of $\pi_k$ with respect to $p$ and equating it to zero yields:

$$\frac{\partial \pi_k}{\partial p} = (1 - x_{WoM}^*) x_{WoM}^* \left(1 - 2^{\alpha^*-1}(A^*)^k\right) + \frac{2^{\alpha^*-2}(1 - x_{WoM}^*) x_{WoM}^*(A^*)^k \log[2]}{(1 - p_{WoM}^*) \log \left[\frac{1}{1-h}\right]} = 0$$

Where $\alpha^* \text{ and } A^*$ are as above. Equating the first term to the second yields:

$$\left(1 - 2^{\alpha^*-1}(A^*)^k\right) = -\frac{2^{\alpha^*-2}(A^*)^k \log[2]}{(1 - p_{WoM}^*) \log \left[\frac{1}{1-h}\right]}$$

Substituting $\left(1 - 2^{\alpha^*-1}(A^*)^k\right)$ into $\frac{\partial \pi_k}{\partial x}$ and solving for $p_{WoM}^*$ yields:

$$p_{WoM}^* = 1 - \frac{(1 - 2x_{WoM}^*) A^* \log[2]}{(2 - \sqrt{2}) k(1 - x_{WoM}^*) x_{WoM}^* \log \left[\frac{1}{1-h}\right]}$$

By Proposition 3, $x_{WoM}^* \in [1/2, 1]$. Moreover, $h \in (1/2, 1)$. Therefore $p_{WoM}^* \in [0, \bar{p}]$.

Step 2. Substituting $p_{WoM}^*$ into the FOC for $x$ yields:

$$\frac{\partial \pi_k}{\partial x} = 2^{\alpha^*-1} \left(1 - \frac{1}{\sqrt{2}}\right) k(1 - x_{WoM}^*) x_{WoM}^*(A^*)^{k-1} + (1 - 2x_{WoM}^*) \left(1 - 2^{\alpha^*-1}(A^*)^k\right) = 0 \quad (10)$$

Where $A^*$ is as above and $\alpha^*$ can be completely expressed in terms of $x_{WoM}^*$ by substituting $p_{WoM}^*$ in $x_{WoM}^*$. To calculate $\frac{dx_{WoM}}{dk}$ we use the IFT. Calculating $\frac{\partial^2 \pi_k}{\partial x^2}$ yields:
\[
\frac{\partial^2 \pi}{\partial x^2} = -2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right)^2 (k - 1) k (1 - x_{WOM}^*) x_{WOM}^* (A^*)^{k-2} - 2^{\alpha} \left( 1 - \frac{1}{\sqrt{2}} \right) k (1 - 2x_{WOM}^*)(A^*)^{k-1} - 2 \left( 1 - 2^{\alpha - 1} (A^*)^k \right) - \frac{2^{\alpha - 1} (2 - \sqrt{2}) k (1 - x_{WOM}^*) x_{WOM}^* (A^*)^k}{2 - (2 - \sqrt{2}) x_{WOM}^*} c_1
\]

Where:

\[
c_1 = \left( -\frac{2}{1 - 2x_{WOM}^*} + \frac{1}{1 - x_{WOM}^*} - \frac{1}{x_{WOM}^*} + \frac{1}{2A^*} \right) \left( 1 - \frac{2 - \sqrt{2}}{2} k (1 - x_{WOM}^*) x_{WOM}^* \right) \left( 1 - \frac{2 - \sqrt{2}}{2} k (1 - x_{WOM}^*) x_{WOM}^* \right)
\]

By Proposition 3, \( x_{WOM}^* \in [1/2, 1] \). As \( h \in (1/2, 1) \) then \( \frac{\partial^2 \pi_k}{\partial x^2} \leq 0 \). Hence, the sign of \( \frac{dx_{WOM}^*}{dk} \) is given by \( \frac{\partial^2 \pi_k}{\partial x \partial k} \). Therefore:

\[
\frac{\partial^2 \pi_k}{\partial x \partial k} = -2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right) (1 - x_{WOM}^*) x_{WOM}^* (A^*)^{k-1} (1 + \log[A^*]) - 2^{\alpha - 1} (1 - 2x_{WOM}^*) (A^*)^k \log[A^*] - 2^{\alpha - 2} \left( 1 - \frac{1}{\sqrt{2}} \right) (2 - \sqrt{2}) k (1 - p_{WOM}^*) (1 - x_{WOM}^*)^2 (x_{WOM}^*)^2 (A^*)^{k-1} \frac{p_{WOM}^* (1 - 2x_{WOM}^*) A^*}{p_{WOM}^* (1 - x_{WOM}^*) x_{WOM}^* (A^*)^{k-1}} + \frac{2^{\alpha - 2} (2 - \sqrt{2}) (1 - p_{WOM}^*) (1 - x_{WOM}^*) x_{WOM}^* (A^*)^{k-1}}{p_{WOM}^*}
\]

Notice that from \( \frac{\partial^2 \pi_k}{\partial x \partial k} \) it is not possible to determine its sign. We, therefore, use equation (8) to simplify \( \frac{\partial^2 \pi_k}{\partial x \partial k} \). Simplifying:

\[
\frac{\partial^2 \pi_k}{\partial x \partial k} = -(1 - 2x_{WOM}^*) \kappa(k, x_{WOM}^*)
\]

Where:

\[
\kappa(k, x_{WOM}^*) = \frac{c_2}{k} + \frac{1}{k} \left( 1 - 2^{\alpha - 1} (A^*)^k \right) + \log[A^*]
\]

and,

\[
c_2 = -2^{\alpha - 2} (A^*)^{k-1} \frac{(2 - \sqrt{2}) k (1 - x_{WOM}^*) x_{WOM}^* (A^*)^k - 2(1 - 2x_{WOM}^*) A^*) \log[2]}{p_{WOM}^* \log \left[ \frac{1 - k}{h} \right]}
\]

Notice that the sign of the numerator for \( \frac{dx_{WOM}^*}{dk} \) is given by \( \kappa \). We nor proceed to show that \( \kappa \) is decreasing in \( k \) and that there exists a \( \hat{k} \) such that \( \kappa(\hat{k}, x_{WOM}^*) = 0 \). Therefore,
Let $k = 1$. Then $\hat{\kappa}(1,x_{\text{WoM}})$ is given by:

$$\hat{\kappa}(1,x_{\text{WoM}}) = c_2 + \left(1 - 2^{a-1}A^*\right) + \log[A^*]$$

As by Proposition 3, $x_{\text{WoM}} \in [1/2, 1]$, then $\hat{\kappa}(1,x_{\text{WoM}}) > 0$. Now, let $k = K$. Then $\hat{\kappa}(K,x_{\text{WoM}})$ is given by:

$$\hat{\kappa}(K,x_{\text{WoM}}) = c_2K + \frac{1}{K} \left(1 - 2^{a-1}(A^*)^K\right) + \log[A^*]$$

When $k = K$, the value of $\hat{\kappa}(K,x_{\text{WoM}})$ is dependent on $K$ and $x_{\text{WoM}}$. Notice, however, that as $K \to \infty$, then $\kappa(K,x_{\text{WoM}}) \to \log[A^*]$. As $x \in [0,1]$, then $\log[A^*] < 0$. This implies that when as $K \to \infty$, then $\kappa(K,x_{\text{WoM}}) < 0$ for any $x \in [0,1]$. We now proceed to show that $\hat{\kappa}$ is decreasing in $k$.

Consider the case for $x_{\text{WoM}} \equiv \bar{x}$. At $k = 1$, the value for $\kappa$ is $\kappa(1,\bar{x}) > 0$. On the other hand, at $k = K$, the value for $\hat{\kappa}$ is $\hat{\kappa}(K,\bar{x}) < 0$ for a sufficient large $K$. Finally, notice that at $\bar{x}$, $\hat{\kappa}$ is monotonically decreasing on $k$. This implies that there exists a $\hat{k}$ such that $\hat{\kappa}(\hat{k},\bar{x}) = 0$. The latter conveys that if $k < \hat{k}$, then $\hat{\kappa} > 0$. Otherwise, if $k > \hat{k}$, then $\hat{\kappa} < 0$. Moreover, as $\hat{\kappa}(1,x) > 0$ for any $x \in [0,1]$ and $\hat{\kappa}(K,x) < 0$ for any $x \in [0,1]$ and a sufficient large $K$, it has to be the case the the argument above holds for any $x_{\text{WoM}}$.

Finally, using the IFT if $k < \hat{k}$ yields $\frac{dx_{\text{WoM}}}{dk} > 0$. If $k > \hat{k}$, yields $\frac{dx_{\text{WoM}}}{dk} < 0$.

**Step 3.** From the value of $p_{\text{WoM}}^*$ in **Step 1**, it follows that $p_{\text{WoM}}^*$ is decreasing in $k$. Details are omitted.

**Step 4.** The result on profits follows directly from result (1) in Proposition 1. Details are omitted. \[\square\]

**Proposition 10.** If $h$ increases, then:

1. Total Welfare increases.
2. Consumer Surplus decreases.

**Proof.** Define welfare as:

$$W = \Pi + CS = pD + \frac{1}{2}D(1-p) + \frac{1}{2}D(-p)$$

Where $D$ is the demand. Simplifying yields:

$$W = \frac{D}{2}$$
From Proposition 2 we know that an increase in \( h \) increases \( D \). Hence \( \frac{dW}{dh} > 0 \). Moreover, Consumer Surplus is defined as follows:

\[
CS = \frac{1}{2} (1 - p) + \frac{1}{2} (-p).
\]

Simplifying yields:

\[
CS = \frac{D}{2} (1 - 2p)
\]

From Proposition 2, we know that \( p \) is increasing in \( h \). As \( p \geq \frac{1}{2} \), then \( \frac{dCS}{dh} < 0 \).

**Proposition 11.** If \( k \) increases, then:

1. Total Welfare increases.
2. Consumer Surplus decreases.

**Proof.** Define welfare as:

\[
W = \Pi + CS = pD + \frac{1}{2} D(1 - p) + \frac{1}{2} D(-p)
\]

Where \( D \) is the demand. Simplifying yields:

\[
W = \frac{D}{2}
\]

From Proposition 1 we know that \( D \) is increasing in \( k \). Hence \( \frac{dW}{dk} > 0 \). Moreover, Consumer Surplus is defined as follows:

\[
CS = \frac{1}{2} (1 - p) + \frac{1}{2} (-p).
\]

Simplifying yields:

\[
CS = \frac{D}{2} (1 - 2p)
\]

From Proposition 1, we know that \( p \) is increasing in \( k \). As \( p \geq \frac{1}{2} \), then \( \frac{dCS}{dk} < 0 \).

**Proposition 12.** The optimal strategy of the monopolist can be either increasing in degree or, first increasing in degree and then decreasing in degree.

**Proof.** We consider in Step 1 the marginal returns for a degree \( s \) individual. Following, we study the relationship between the marginal returns for a degree \( s \) individual and the marginal returns for a degree \( s' \) individual, for any \( s' > s \). We note that marginal returns between different degree individuals differ by a function we call \( f \). In Step 2, we develop \( f \) to determine how does it change with \( s \). In Step 3, we identify two cases for \( f \) and derive, for each of them, the relationship between \( x^*_s \) and \( x^*_{s+1} \).
Step 1. Note that for all $s \in O$ the marginal returns to degree are:

$$
\frac{d\Pi(x \mid P)}{dx_s} = P(s)p \left[ -\Theta(x) + \sum_k P(k)(1 - x_k)f(s, x_s) \right]
$$

Where $f(s, x_s)$ is given by:

$$
f(s, x_s) = 1 - 2^{\alpha - 1} \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right)x_s \right)^s + 2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right)sx_s \left( 1 - \left( 1 - \frac{1}{\sqrt{2}} \right)x_s \right)^{s-1}
$$

Within brackets, the only term that differs across $s$ is $f(s, x_s)$. Assume there exists an $s$ and an $x_s^*$ so that $\frac{d\Pi(x^* \mid P)}{dx_s} = 0$. It then follows that if $f$ is increasing in $s$, then $\frac{d\Pi(x^* \mid P)}{dx_s} > 0$ at $x_{s'} = x_s^*$ for every $s' \geq s$. On the other hand, if $f$ is decreasing in $s$, then $\frac{d\Pi(x^* \mid P)}{dx_s} < 0$ at $x_{s'} = x_s^*$ for every $s' \geq s$.

Step 2. Call:

$$
A = 1 - \left( 1 - \frac{1}{\sqrt{2}} \right)x_s
$$

Then:

$$
f(s, x_s, A) = 1 - 2^{\alpha - 1}A^s + 2^{\alpha - 1} \left( 1 - \frac{1}{\sqrt{2}} \right)sx_sA^{s-1}
$$

Taking its derivative with respect to $s$ yields:

$$
\frac{df}{ds} = B\psi(s, x_s, A)
$$

Where:

$$
\psi(s, x_s, A) = \log \left\{ A \left[ 2 - (s + 1) \left( 2 - \sqrt{2} \right)x_s \right] - \left( 2 - \sqrt{2} \right)x_s \right\}
$$

and,

$$
B = -\frac{2^{\alpha - 1}A^s}{2 - (2 - \sqrt{2})x_s}
$$

As $x_s \in [0, 1]$, then $B < 0$. Therefore, the sign of $\frac{df}{ds}$ depends on the sign of $\psi$.

Step 3. Note that $\psi(s, x_s, A)$ is monotonically increasing on $s$. We now proceed to evaluate the value of $\psi$ at $s = 1$ and $s = K$ to determine the sign of $\psi$.

Let $s = 1$. Then $\psi$ is given by:

$$
\psi(1, x_s, A) = \log \left\{ A \left[ 2 - 2 \left( 2 - \sqrt{2} \right)x_s \right] - \left( 2 - \sqrt{2} \right)x_s \right\}
$$

In this case, $\psi(1, x_s, A) < 0$ for any $x_s \in [0, 1]$. Now, let $s = K$. Then $\psi$ is given by:

$$
\psi(K, x_s, A) = \log \left\{ A \left[ 2 - (K + 1) \left( 2 - \sqrt{2} \right)x_s \right] - \left( 2 - \sqrt{2} \right)x_s \right\}
$$
In this case, there exists an $x_s$ such that $\psi(K, x_s, A) \leq 0$ for every $x_s \in [0, \bar{x_s}]$ and $\psi(K, x_s, A) > 0$ for every $x_s \in (\bar{x_s}, 1]$. Monotonicity of $\psi(s, x_s, A)$ implies that $\psi$ is either negative for all its domain, or first negative and then positive. We now identify two cases.

**Case 1.** Suppose there exists an $x_{K-1}^* > 0$ such that $\frac{d\Pi(x^*|P)}{dx_{K-1}} = 0$ and $\psi(K, x_{K-1}^*, A) \leq 0$. As $\psi(1, x_{K-1}^*, A) < 0$ for any $s \in O$, then, monotonicity of $\psi$ on $s$ implies that $\frac{d\Pi(x^*_s)}{ds} > 0$ for every $s \in O \setminus \{K\}$. Concavity of $\Pi$ on $x_s$ conveys $x_{s+1}^* > x_s^*$.

**Case 2.** Suppose there exists an $x_{K-1}^* > 0$ such that $\frac{d\Pi(x^*|P)}{dx_{K-1}} = 0$ and $\psi(K, x_{K-1}^*, A) > 0$. As $\psi(1, x_{K-1}^*, A) < 0$ for any $s \in O$, then, there exists an $\bar{s}$ such that $\psi(\bar{s}, x_{K-1}^*, A) = 0$. Define $O' \subset O$ as the set containing the groups $O' = \{1, 2, ..., \bar{s}\}$. Then, monotonicity of $\psi$ on $s$ implies that $\frac{d\Pi(x^*_s)}{ds} > 0$ for every $s \in O'$ and concavity of $\pi_k$ on $x_s$ conveys $x_{s+1}^* > x_s^*$ for every $s \in O'$. On the other hand, monotonicity of $\psi$ on $s$ implies that $\frac{d\Pi(x^*_s)}{ds} < 0$ for every $s \in O \setminus O'$. Finally, concavity of $\pi_k$ on $x_s$ conveys $x_{s+1}^* < x_s^*$ for every $s \in O \setminus O'$.

**Proposition 13.** Let $P'$ be a Mean Preserving Spread of $P$. Then, optimal profits under $P'$ are lower than under $P$.

**Proof.** To prove Proposition 5, in Step 1 we prove the returns to degree for $\Pi(x | P)$ are concave for any $x_l \in [0, 1]$ for every $l \in O$. In Step 2, we draw conclusions on the profit level $\Pi(x_P | P)$ and $\Pi(x_{P'} | P')$.

**Step 1.** To check for concavity of returns to degree of $\Pi$, it suffices to check for $\frac{\partial^2 \Theta}{\partial x^2}$. Taking the second derivative of $\Theta(x)$ with respect to $l$ yields:

$$\frac{\partial^2 \Theta}{\partial x^2} = -\sum_l P_l x_l 2^{x_l-1} A^l \log |A|^2$$

Where $\alpha$ is as in equation (3) and $A$ as in equation (10). As $x_l \in [0, 1]$, then $0 \leq A \leq 1$. Therefore $\frac{\partial^2 \Theta}{\partial x^2} \leq 0$. It, then, follows that $\frac{\partial^2 \Pi}{\partial x^2} \leq 0$.

**Step 2.** In turn, Step 1 implies that:

$$\Pi(x_{P'} | P') \leq \Pi(x_P | P) \leq \Pi(x_{P} | P)$$

Where the first inequality follows because $\Pi$ is concave in degree and $P'$ is a MPS of $P$. The second inequality follows from optimality of $x_P$ under $P$. □

**References**


